THERMAL BOUNDARY CONDITIONS FOR WATER FLOW WITH MOVING BOUNDARIES

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By EDWARD R. HOLLEY and NOBUHIRO YOTSUKURA

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THERMAL BOUNDARY CONDITIONS FOR WATER FLOW WITH MOVING BOUNDARIES

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ABSTRACT

General thermal boundary conditions are derived for the free surface and the bed for unsteady open-channel flow where both boundaries may be in motion. Since any movement of the bottom boundary normally involves sediment movement, it is assumed in the analysis that the water is carrying suspended sediment. The boundary conditions consider a variety of radiative, advective, and diffusive fluxes of heat including the possiblity of some solar radiation reaching the stream bed. Kinematical boundary conditions for the water, the sediment, and the suspension have been presented since they can be used to simplify the thermal boundary conditions. Application of the boundary conditions for calculation of vertical distributions of temperature requires knowledge of the diffusive transports of heat immediately below the free surface and immediately above the bed. There are some aspects of these diffusive transports which are not well defined.

INTRODUCTION

Considerable attention has been given to various mechanisms of heat transfer at a free water surface (Anderson, 1954; Ryan and Harleman, 1973) and to temperature distributions in water bodies under both natural conditions (Edinger and Geyer, 1965; Morse, 1970) and conditions of thermal loading

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(Paily, Macagno and Kennedy, 1974). Situations have been considered both with vertical uniformity of temperature (Harleman, 1972; Jobson and Yotsukura, 1973) and with vertical distributions of temperature (Schiller and Sayre, 1975; Prych and others, 1976). The references cited are just a few typical examples and are not intended to provide an exhaustive list.

In some problems, it is beneficial to solve the differential thermal-energy conservation equation to obtain information on the spatial and(or) temporal temperature distributions. Regardless of whether the solution of the differential equation is obtained analytically or numerically, appropriate boundary conditions must be used. When solving for vertically nonuniform distributions of water temperature, thermal boundary conditions must be satisfied at both the free surface and the channel bottom. For problems dealing with depth-averaged or cross-section-averaged temperatures, energy fluxes associated with thermal boundary conditions appear as source and sink terms in the thermal-energy conservation equation.

The customary formulation of thermal boundary conditions for problems with vertically non-uniform temperature distributions has been to assume that the only flux normal to the boundary in the water is a diffusive flux which is the product of the temperature gradient and the diffusion coefficient at the boundary. Therefore, the thermal boundary condition has normally been written by equating this diffusive flux to the net thermal energy flux across the boundary. This approach, which is analogous to that for the conduction of heat in solids, is straightforward for steady flows. However, it is not immediately obvious that the same formulation applies when the flow is unsteady and the boundaries are in motion, because then advective thermal

transport at the boundaries has to be considered. Chen (1971) presented a type of thermal boundary condition for moving boundaries, but he used the customary formulation without reference to the advective transport, and his presentation neglected heat transfer associated with radiation and latent heat of vaporization, both of which are normally quite significant at the air-water interface.

The purpose of this paper is to present general thermal boundary conditions for unsteady, free-surface water flow with moving boundaries. In an effort to be general in the analysis, several terms have been included which may be negligible in some situations. Since any movement of the bottom boundary normally involves sediment movement, it is assumed in the analysis that the water is carrying suspended sediment. Precipitation, evaporation, and infiltration are included to account for water inflows and outflows and the associated heat fluxes. The possibility of solar radiation reaching the stream bed is also considered. It appears that solar radiation can provide an important heat input at the bed, especially for streams which are relatively shallow and(or) in which the water is relatively clear.

First the kinematical boundary conditions are derived for the water, for the sediment, and for the water-sediment mixture. The general thermal boundary conditions are then presented in their complete forms as well as in reduced forms which can be obtained by use of the kinematical boundary conditions. These are the boundary conditions for solving differential thermal-energy conservation equations to obtain spatial and temporal variations of temperature in a water body.

DEFINITIONS

Consider an unsteady free surface flow as depicted in figure 1. Let $y = S(x,z,t)$ describe the unsteady free surface and $y = B(x,z,t)$ describe the unsteady bottom surface. Then $h(x,z,t) = S-B$ is the depth of water. Let P = precipitation rate, E = evaporation rate, and I = infiltration rate, with each one being defined as an intensity or volume flux per unit area and having the dimensions of length per time. The positive directions for P, E, and I are indicated by arrows in figure 1. It is assumed that P is a rate per unit of horizontal area while E is per unit of surface area and I is per unit of bottom area.

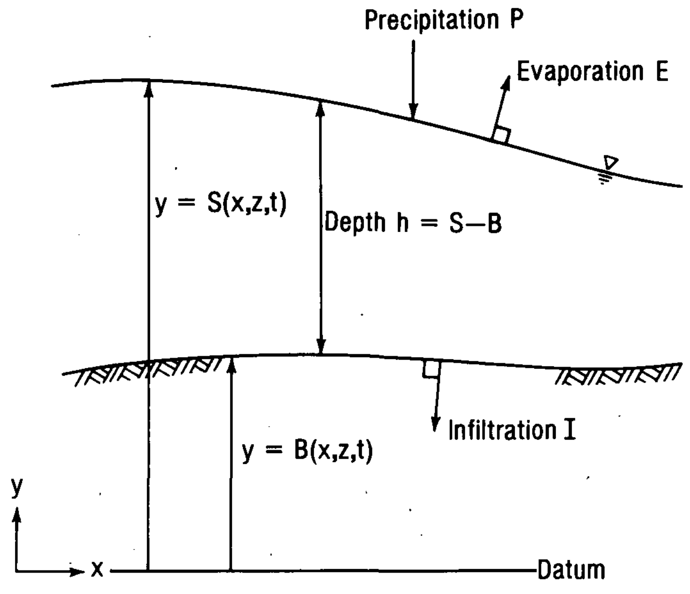

Figure 1.--Longitudinal profile of a free surface flow.

The water is assumed to be carrying suspended sediment. The combined water and sediment will be called the suspension. Some considerations will relate to the suspension, some to the sediment, and some just to the water in the suspension. The following definitions will be used:

ρ = density of suspension (mass per unit volume)

c_s = mass of sediment per unit volume of suspension

ρ_s = density of sediment particles (mass per unit volume)

c_s/ρ_s = volume of sediment per unit volume of suspension

c_w = mass of water per unit volume of suspension

ρ_w = density of water (mass per unit volume)

c_w/ρ_w = volume of water per unit volume of suspension

x,y,z = Cartesian coordinates with x and z being horizontal

$\vec{i},\vec{j},\vec{k}$ = unit vectors in x,y,z directions

u,v,w = velocity components in x,y,z directions

$\vec{V}$ = velocity vector, equation 1

e_x,e_y,e_z = diffusion coefficients in x,y,z directions

$\vec{D}$ = diffusive flux vector operator, equation 2

θ = porosity of bed material

$\vec{n}$ = unit outward normal vector at boundary

σ = subscript referring to water surface

β = subscript referring to bottom boundary

s = subscript referring to the sediment

w = subscript referring to the water

The velocity vector and the diffusive flux operator listed above are defined as

$$\vec{V} = \vec{i}u + \vec{j}v + \vec{k}w \qquad (1)$$

and

$$\vec{D} = -\vec{i}e_x \frac{\partial}{\partial x} - \vec{j}e_y \frac{\partial}{\partial y} - \vec{k}e_z \frac{\partial}{\partial z} \qquad (2)$$

Note that the above definitions for velocity and diffusion terms, including equations 1 and 2, apply not only to the suspension but also to the sediment and the water. The omission of subscripts, s and w, on these terms is to avoid unnecessary duplications. In the following sections of the paper, however, the absence of s or w as a subscript on a velocity or diffusion term will indicate that the term refers to the suspension.

The relation between the velocities of the sediment, water, and suspension can be obtained by considering the flux through an incremental area $\Delta\alpha$. The fraction of the area cutting through sediment particles on the average is $\Delta\alpha_s/\Delta\alpha = c_s/\rho_s$, and the fraction cutting through water is $\Delta\alpha_w/\Delta\alpha = c_w/\rho_w$, since, on the average, the cross-sectional area ratio of sediment particles to water is the same as the ratio of sediment volume to water volume. The suspension velocity, $\vec{V}$, will be defined as the volume average velocity through $\Delta\alpha$, so that

$$\vec{V} = \frac{\vec{V}_s \Delta\alpha_s}{\Delta\alpha} + \frac{\vec{V}_w \Delta\alpha_w}{\Delta\alpha} = \frac{\vec{V}_s c_s}{\rho_s} + \frac{\vec{V}_w c_w}{\rho_w} \qquad (3)$$

Some aspects of the following discussion are contingent on this definition of $\vec{V}$. If $\vec{n}$ is the unit vector normal to $\Delta\alpha$, then $\vec{n}\cdot\vec{V}\,\Delta\alpha$ is the volumetric flux of suspension through $\Delta\alpha$.

From the definitions of the densities and concentrations, it can be shown that

$$\frac{c_s}{\rho_s} + \frac{c_w}{\rho_w} = 1, \qquad (4)$$

since the sum of the sediment volume and water volume must equal the total volume.

The analyses for developing the boundary conditions will relate to an element, $d\sigma$, of free surface area which has a horizontal projected area of $dA = dxdz$ and to a similar element, $d\beta$, of channel bottom area. The relations between $d\sigma$ and dA and between $d\beta$ and dA are (Sokolnikoff and Redheffer, 1958)

$$d\sigma = \sqrt{1 + \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2}\; dA \qquad (5)$$

$$d\beta = \sqrt{1 + \left(\frac{\partial B}{\partial x}\right)^2 + \left(\frac{\partial B}{\partial z}\right)^2}\; dA \qquad (6)$$

The unit outward normal vectors for the free surface, $\vec{n}_\sigma$, and for the bed, $\vec{n}_\beta$, can be written as

$$\vec{n}_\sigma = \left(-\vec{i}\,\frac{\partial S}{\partial x} + \vec{j} - \vec{k}\,\frac{\partial S}{\partial z}\right) \frac{dA}{d\sigma} \qquad (7)$$

and

$$\vec{n}_\beta = \left(\vec{i}\,\frac{\partial B}{\partial x} - \vec{j} + \vec{k}\,\frac{\partial B}{\partial z}\right) \frac{dA}{d\beta} \qquad (8)$$

KINEMATICAL BOUNDARY CONDITION AT FREE SURFACE

The kinematical boundary conditions for the water and for the sediment can be obtained in a number of ways. The method used here was selected because it has a rather straightforward physical interpretation. Consider the situation shown in figure 2, where S_1 is a segment of the free surface at time t and S_2 is the free surface at an increment of time dt later. Let $d\lambda$ be the incremental volume between S_1 and S_2. Then

$$\frac{\partial \lambda}{\partial t} = \frac{\partial S}{\partial t}\, dA. \qquad (9)$$

Continuity requires that the instantaneous time rate of change of mass in the incremental volume element $d\lambda$, which is created by the moving water surface, must equal the mass rate of inflow normal to $d\sigma$, which is an element of the initial S_1, plus other mass rates of inflow across the moving S. This concept can be applied both to the sediment and to the water to obtain boundary conditions for each one, and then the results can be added to obtain the boundary condition for the suspension. Consideration of mass fluxes across the vertical sides of $d\lambda$ gives rise only to second-order terms which are negligible relative to the mass fluxes across $d\sigma$ and S.

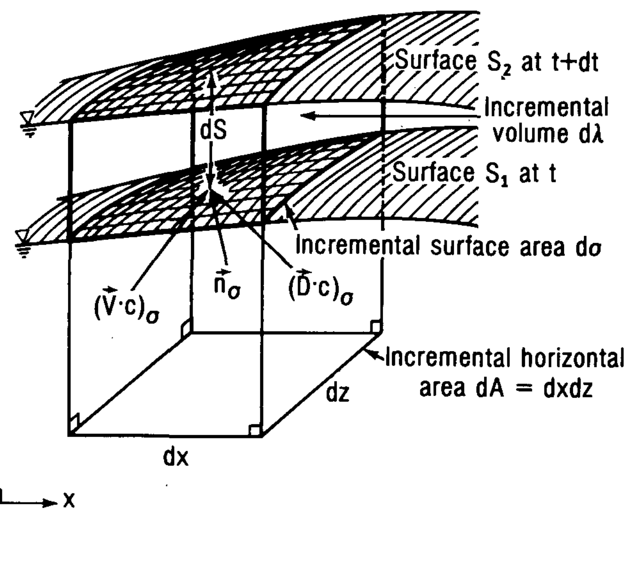

Figure 2.--Element of a moving free surface.

Sediment

The mass flux of sediment normal to $d\sigma$ is a result of advection and diffusion and can be written as the advective flux, $(\vec{n}\cdot\vec{V}_s c_s)_\sigma d\sigma$, plus the diffusive flux, $(\vec{n}\cdot\vec{D}_s c_s)_\sigma d\sigma$. The rate at which sediment is going into storage in $d\lambda$ is the surface concentration of sediment, $c_{s\sigma}$, times $(\partial S/\partial t)dA$, which is the rate at which volume is being created due to the movement of S. Since there are no fluxes of sediment across S, the boundary condition can be written as

$$(\vec{n}\cdot\vec{V}_s c_s + \vec{n}\cdot\vec{D}_s c_s)_\sigma d\sigma = c_{s\sigma}\frac{\partial S}{\partial t}dA \qquad (10)$$

It will be assumed that terms like the product $(\partial c_s/\partial t)_\sigma(\partial S/\partial t)$ are negligible second-order terms. In most of this paper, the fluxes will be written in vector form, as in equation 10. However, for this one equation, the vectors will be expanded into a Cartesian coordinate system for illustrative purposes. Using equations 1, 2, and 7, equation 10 can be expanded to

$$(-u_{s\sigma}\frac{\partial S}{\partial x} + v_{s\sigma} - w_{s\sigma}\frac{\partial S}{\partial z})c_{s\sigma} + (e_{xs}\frac{\partial c_s}{\partial x})_\sigma\frac{\partial S}{\partial x} - (e_{ys}\frac{\partial c_s}{\partial y})_\sigma + (e_{zs}\frac{\partial c_s}{\partial z})_\sigma\frac{\partial S}{\partial z}$$

$$= c_{s\sigma}\frac{\partial S}{\partial t} \qquad (11)$$

Water

The mass flux of water across $d\sigma$ and the rate of change of mass storage of water in $d\lambda$ can be written by analogy to the similar terms for the sediment. However, in addition to the type of terms which were considered for the sediment, there are fluxes of water across S due to evaporation and precipitation, as indicated in figure 1. Thus, the boundary condition for the water can be written as

$$\rho_w PdA - \rho_w Ed\sigma + (\vec{n}\cdot\vec{V}_w c_w + \vec{n}\cdot\vec{D}_w c_w)_\sigma \, d\sigma = c_{w\sigma} \frac{\partial S}{\partial t} dA \qquad (12)$$

Suspension

Equations 10 and 12 give the boundary conditions for the sediment and for the water. These equations can be combined to give the boundary conditions for the suspension. First, however, divide equation 10 by ρ_s and equation 12 by ρ_w. This converts the equations from mass flux to volume flux. Assume that both the sediment particles and the water are incompressible so that ρ_s and ρ_w are both constant. Adding the two equations and using equations 3 and 4, the result can be written as

$$PdA - Ed\sigma + (\vec{n}\cdot\vec{V})_\sigma \, d\sigma = \frac{\partial S}{\partial t} dA, \qquad (13)$$

where $\vec{V}$ is the volume average velocity expressed by equation 3. Equation 13 gives the kinematical boundary condition at the water surface for the suspension.

Equation 13 does not contain any diffusive flux terms. This is based on the fact that the definition of $\vec{V}$ implies that the net diffusive flux of suspension volume relative to $\vec{V}$ must be zero, or

$$\frac{\vec{D}_s c_s}{\rho_s} + \frac{\vec{D}_w c_w}{\rho_w} = 0 \qquad (14)$$

Equation 14 may be interpreted as saying that the diffusive flux of sediment volume in one direction relative to $\vec{V}$ must be compensated by the diffusive flux of water volume in the opposite direction. A discussion of various ways of defining average velocities and diffusive fluxes relative to those velocities is given by Bird and others (1960).

KINEMATICAL BOUNDARY CONDITION AT CHANNEL BOTTOM

Considerations for the bottom boundary are similar in many respects to those for the free surface. However, for the bottom boundary to move vertically, there must be a movement of sediment due to entrainment or deposition. Any bedload movement relates primarily to longitudinal movement of sediment and not to vertical movement of the bed. In other words, the bedload movement gives rise to transport through the vertical sides of the incremental volume element and therefore gives rise to negligible second-order terms, as was mentioned in connection with the kinematical condition at the free surface.

A moving bottom boundary is illustrated in figure 3, where B_1 is the position of the bottom boundary at time t and B_2 is at time $t + dt$. It is assumed that, at time t, the volume $d\lambda'$ between B_1 and B_2 is occupied by a volume $\theta d\lambda'$ of water and $(1-\theta)d\lambda'$ of bed material, where θ is the porosity of the bed material. It is also assumed that, at time $t + dt$ when the bed is at B_2, $d\lambda'$ is occuiped by a volume $(c_{w\beta}/\rho_w)d\lambda'$ of water and $(c_{s\beta}/\rho_s)d\lambda'$ of suspended sediment. Although figure 3 and the preceding discussion are presented in terms of bed erosion, the results given in this section can also be shown to be applicable for the case of deposition. By analogy to the free surface equations and utilizing figure 3 and the associated discussion for writing the rates at which sediment and water are going into storage in $d\lambda'$, the boundary conditions for the channel bottom can be expressed in eqs. 15, 16, and 17 in the following discussion.

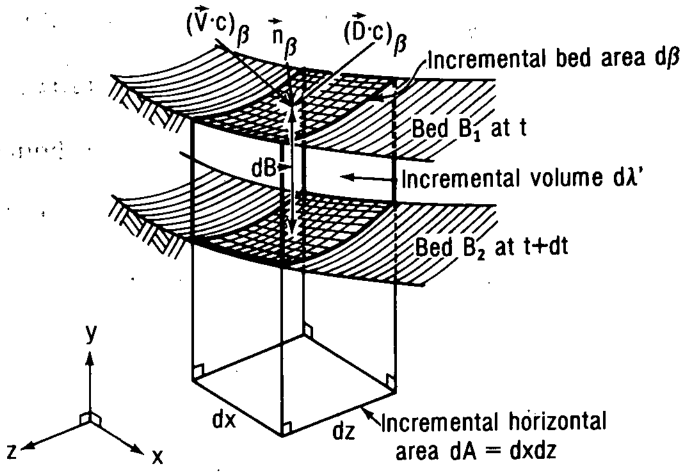

Figure 3.--Element of a moving channel bed.

Sediment

$$(\vec{n}\cdot\vec{V}_s c_s + \vec{n}\cdot\vec{D}_s c_s)_\beta d\beta = \left[\rho_s(1-\theta) - c_{s\beta}\right]\frac{\partial B}{\partial t}\, dA, \quad (15)$$

Water

$$-\rho_w I d\beta + (\vec{n}\cdot\vec{V}_w c_w + \vec{n}\cdot\vec{D}_w c_w)_\beta d\beta = (\rho_w\theta - c_{w\beta})\,\frac{\partial B}{\partial t}\, dA, \quad (16)$$

where $-\rho_w I d\beta$ is the flux across B due to infiltration as defined in figure 1. Accretion from ground water can be represented by a negative value of I.

Suspension

Dividing equation 15 by ρ_s and equation 16 by ρ_w, adding the resulting equations, and using equations 3, 4, and 14, the boundary condition for the suspension is

$$-I d\beta + (\vec{n}\cdot\vec{V})_\beta d\beta = 0 \quad (17)$$

Recall that $\vec{V}$ in equation 17 is the volume average velocity expressed by equation 3.

HEAT TRANSFER MECHANISMS AT FREE SURFACE

Various heat transfer mechanisms at the free surface have been considered in detail in a number of references (for example, Ryan and Harleman, 1973). The only considerations presented here are those needed for obtaining the appropriate thermal boundary conditions. In general, let ϕ be the heat flux per unit area. Specifically, the heat flux associated with the various mechanisms will be identified as follows:

Symbol	Heat flux associated with
ϕ_P	= heat content of precipitation
ϕ_E	= heat content of evaporated water
ϕ_s	= heat content of sediment being advected and diffused
ϕ_w	= heat content of water being advected and diffused
ϕ_{si}	= incident solar (short wave) radiation
ϕ_{sr}	= reflected solar radiation
ϕ_{ai}	= incident atmospheric (long wave) radiation
ϕ_{ar}	= reflected atmospheric radiation
ϕ_r	= net radiation input at free surface = $(\phi_{si} - \phi_{sr} + \phi_{ai} - \phi_{ar})$
ϕ_{st}	= transmitted solar radiation
ϕ_b	= back radiation from water surface
ϕ_e	= latent heat of vaporization
ϕ_c	= heat conducted from the free surface to air
ϕ_d	= diffusion of heat below the free surface

The time rate of change of heat stored in $d\lambda$ of the suspension per unit area will be labeled ϕ_λ. The terms ϕ_P, ϕ_{si}, ϕ_{sr}, ϕ_{ai}, ϕ_{ar}, ϕ_r, ϕ_{st}, and ϕ_λ are defined per unit horizontal area, while the remainder of the ϕ terms are defined per unit of surface area. Figure 4 depicts the various fluxes, with the positive direction indicated by the arrows. With these definitions, $\phi_r - \phi_{st}$ is the rate of radiation absorption at the free surface itself.

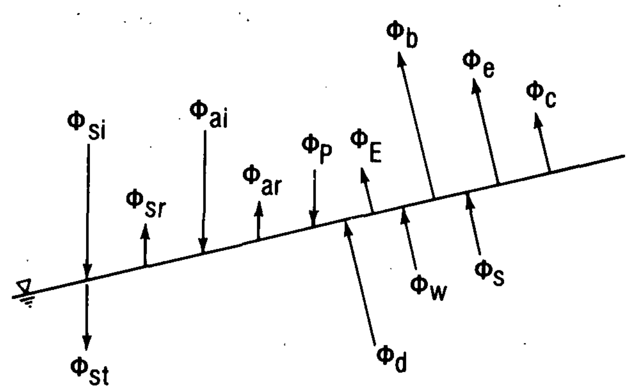

Figure 4.--Heat fluxes at free surface.

The terms ϕ_P, ϕ_E, ϕ_s, ϕ_w, and ϕ_λ relate to heat contained in water, sediment, or suspension; for these, it is convenient to define the heat content relative to an arbitrary reference temperature, T_r, and to assume that temperature differences involved in the ϕ's are small enough that the specific heat and the density are approximately constant. Then, the heat content, H_c, per unit volume becomes

$$H_c = \int_{T_r}^{T} \rho C_p dT = \rho C_p (T - T_r), \qquad (18)$$

where T = temperature and C_p = specific heat. With appropriate subscripts, equation 18 can be applied to the sediment, the water, or the suspension.

It will be assumed that the water and sediment in any elemental volume are at the same temperature. If C_{ps} is the specific heat for the sediment and C_{pw} is for the water, the specific heat of the suspension is

$$C_p = \frac{c_s}{\rho} C_{ps} + \frac{c_w}{\rho} C_{pw} \qquad (19)$$

With these definitions and assumptions, together with T_σ = water surface temperature and T_P = temperature of precipitation, the above ϕ terms can be expressed by following equations.

$$\phi_P = P\rho_w C_{pw}(T_P - T_r) \qquad (20)$$

$$\phi_E = E\rho_w C_{pw}(T_\sigma - T_r) \qquad (21)$$

$$\phi_s = (\vec{n}\cdot\vec{V}_s c_s + \vec{n}\cdot\vec{D}_s c_s)_\sigma C_{ps}(T_\sigma - T_r) \qquad (22)$$

$$\phi_w = (\vec{n}\cdot\vec{V}_w c_w + \vec{n}\cdot\vec{D}_w c_w)_\sigma \, C_{pw}(T_\sigma - T_r) \qquad (23)$$

$$\phi_\lambda = \frac{\partial S}{\partial t} \rho C_p (T_\sigma - T_r) \qquad (24)$$

Note that equations 20 through 24 all express heat fluxes or storage per unit area per unit time.

For the remainder of the ϕ terms, namely, ϕ_{si} through ϕ_d, expressions are available from the literature and will not be discussed here, except for the last term, ϕ_d. The diffusive flux, ϕ_d, is dependent on the thermal diffusivity and the temperature gradient immediately below the free surface and is analogous to the molecular conduction of heat in solids. The expression for ϕ_d can be written as

$$\phi_d = (\vec{n}\cdot\vec{D}_T T)_\sigma \, \rho C_p, \qquad (25)$$

where $\vec{D}_T$ is the thermal equivalent of equation 2 and where it has been assumed that spatial derivatives of ρC_p are negligible. The flux ϕ_d is entirely separate from the heat fluxes, ϕ_s and ϕ_w, which are dependent on the advective and diffusive fluxes of c_s and c_w, respectively.

THERMAL BOUNDARY CONDITIONS AT FREE SURFACE

General representation

The thermal boundary condition at the free surface can be written by analogy to the kinematical boundary conditions. Considering the situation shown in figures 2 and 4, the conservation of heat energy requires that the sum of heat fluxes across the original $d\sigma$ and across the moving S from both above and below must equal the time rate of change of storage of heat in $d\lambda$. Thus, the complete form of the boundary condition at the free surface is given by

$$(\phi_r-\phi_{st})dA + \phi_P dA - \phi_E d\sigma + (\phi_s+\phi_w)d\sigma - (\phi_b+\phi_e+\phi_c)d\sigma + \phi_d d\sigma = \phi_\lambda dA \quad (26)$$

Equation 26 can be reduced by use of the kinematical boundary conditions derived earlier. First multiply equation 10 by $C_{ps}(T_\sigma - T_r)$ and equation 12 by $C_{pw}(T_\sigma - T_r)$. Then add these two equations and substitute equations 19 throught 24. The result is

$$\phi_P \frac{T_\sigma - T_r}{T_P - T_r} dA - \phi_E d\sigma + (\phi_s + \phi_w)d\sigma = \phi_\lambda dA \quad (27)$$

Equation 27 shows the relation among ϕ_P, ϕ_E, ϕ_s, ϕ_w and ϕ_λ , which is dictated by the mass continuity of water and sediment. The reduced form of the thermal boundary condition at the free surface is obtained by subtracting equation 27 from equation 26 and utilizing equation 20:

$$\phi_d = -(\phi_r - \phi_{st})\frac{dA}{d\sigma} - \rho_w C_{pw} P(T_P - T_\sigma)\frac{dA}{d\sigma} + \phi_b + \phi_e + \phi_c \quad (28)$$

Equations 26 and 28, though representing the same boundary condition, often have different applications. The complete form given in equation 26 is normally the most convenient form of the surface boundary condition when the three-dimensional thermal energy equation is being integrated with respect to local depth to obtain the depth-averaged thermal energy equation (Jobson and Yotsukura, 1973). It can be shown that the ϕ_r, ϕ_P, ϕ_E, ϕ_b, ϕ_e, and ϕ_c terms remain as sources and sinks in a depth-averaged equation, while the ϕ_s, ϕ_w, ϕ_d, ϕ_{st}, and ϕ_λ terms are eliminated in the depth-averaging process. Note that, when the flow does not contain sediment $(c_s=0)$, not only is $\phi_s = 0$ but also the diffusive term in ϕ_w is zero because C_w is constant according to equation 4.

When the thermal energy equation is solved for describing vertically nonuniform distributions of temperature, the reduced form of the free surface boundary condition as given in equation 28 is normally convenient to use. Equation 28 does not contain the ϕ_E, ϕ_s, ϕ_w, and ϕ_λ terms. In other words, these fluxes associated with the heat content of moving water and sediment and the heat storage change due to $\partial S/\partial t$ have no bearing on this form of the surface boundary condition by virtue of equation 27. Note also that equation 28 is free of the arbitrary reference temperature, T_r.

The derivation of equation 28 establishes that the customary formulation of thermal boundary condition in terms of surface temperature gradient is indeed correct even for unsteady flows with moving free surfaces. This gradient is contained in ϕ_d in equation 28. See also equations 25 and 2. The derivation, at the same time, indicates that a customary assumption of neglecting ϕ_E is redundant. Equation 28 specifies a relation between T_σ and ϕ_d, as given by equation 25, and is, thus, classified as a boundary condition of the Cauchy type. The dependence on T_σ comes primarily from the fact that the heat fluxes associated with back radiation, ϕ_b, latent heat of vaporization, ϕ_e, and conduction, ϕ_c, are all functions of T_σ. The net incoming radiation, ϕ_r, is independent of T_σ and is normally the most important flux during daylight hours. The transmitted solar radiation, ϕ_{st}, is considered to be independent of T_σ, and the heat flux of precipitation, ϕ_p, is often negiligible.

The gradient diffusion term, ϕ_d, in equation 28 involves the thermal diffusion coefficient normal to the free surface by virture of $\vec{n}\cdot\vec{D}_T$ in equation 25, and this coefficient is normally different from turbulent diffusion coefficients in the main region of flow. Physical conditions surrounding the heat transfer near the free surface require further consideration as given in the next section.

Surface Film

There is a thin region, normally on the order of a millimeter in thickness, immediately below the free surface where there is a steep temperature gradient as illustrated in figure 5. Within the thickness, δ_h, of the region, the temperature varies from T_σ to T_f, where T_σ may be higher or lower than T_f, depending on whether the free surface is being heated or cooled. Miller and others (1975) indicated that the difference between T_σ and T_f is normally less than 2°C. This region with steep gradient has been called a film or skin (Ewing and McAlister, 1960; Holley, 1973). Actually it is a thermal boundary layer and exists primarily because the effective thermal diffusion coefficient, ε_{hf}, in the film normal to the free surface is much less than the turbulent thermal coefficient which is typical below the film.

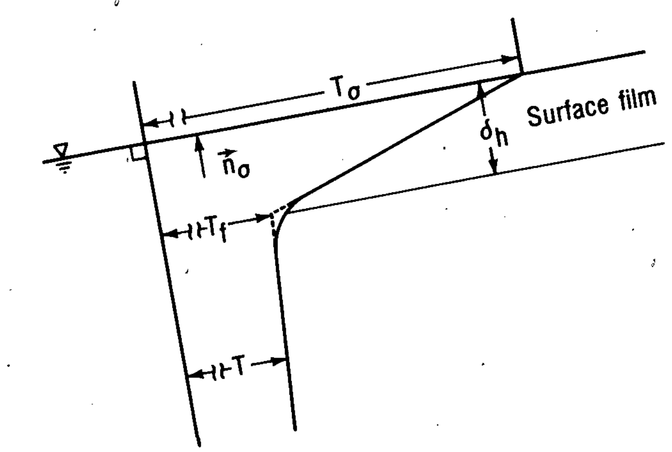

Figure 5.--Temperature distribution below free surface.

The existence of a surface film is well documented. Miller and others (1975) reviewed the literature related to five laboratory studies and seven field studies on determination of the temperature differences across the film. They also made radiometer measurements in the laboratory in the absence of incident shortwave radiation to investigate effects of wind and waves on the temperature difference across the film and on the heat transfer at the free surface. In several of these studies, the "surface" temperatures were obtained with radiometers which had nominal measurement depths up to 0.14 mm, and the "surface" temperatures were compared with the temperature at a depth of a few centimeters in order to determine the temperature differnce across the film.

Holley (1973), using a 0.22-mm diameter thermistor probe, found linear temperature distributions in the surface film in a mixing tank, where agitation was produced by vertically oscillating grids. For his meaurements,

δ_h ranged between 3 and 6 mm, and ε_{hf}, which was constant for each test, ranged from one to eight times the molecular thermal conductivity, depending on the intensity of grid mixing. For natural mixing, it is frequently assumed that ε_{hf} equals the molecular thermal conductivity. Miller and others (1975) pointed out that a linear gradient in the film is characteristics of diffusive transport, while vertical convective transport would produce a logarithmic temperature distribution.

Assuming diffusive transport in the film and a constant scaler thermal diffusion coefficient, ε_{hf}, in the film normal to the surface, equation 28 may be written as

$$\varepsilon_{hf}\rho C_p \left(\frac{\partial T}{\partial n}\right)_\sigma = (\phi_r - \phi_{st})\frac{dA}{d\sigma} + \rho_w C_{pw} P(T_p - T_\sigma)\frac{dA}{d\sigma} - \phi_b - \phi_e - \phi_c \qquad (29)$$

where n is the outward normal coordinate to the free surface, as shown in figure 5. Further assuming a linear temperature variation in the film, the gradient in the left-hand side of equation 29 may be equated to the total gradient through the film, so that

$$\varepsilon_{hf}\rho C_p \left(\frac{\partial T}{\partial n}\right)_\sigma = \varepsilon_{hf}\rho C_p \frac{T_\sigma - T_f}{\delta_h} \qquad (30)$$

Also, because the heat flux must be continuous at the bottom of the film, it follows that

$$\varepsilon_{hf}\rho C_p \frac{T_\sigma - T_f}{\delta_h} = \varepsilon_{ht}\rho C_p \left(\frac{\partial T}{\partial n}\right)_t, \qquad (31)$$

where ε_{ht} is the turbulent thermal diffusion coefficient and $(\partial T/\partial n)_t$ indicates the temperature gradient, both immediately below the film. It is possible to define

$$K_{hf} = \frac{\varepsilon_{hf}}{\delta_h}, \qquad (32)$$

where K_{hf} is a heat transfer coefficient in the surface film and combines ε_{hf} and δ_h into one parameter. The diffusive flux in the left-hand sides of equations 29 and 31 could then be written in terms of K_{hf}.

The use of equation 29 in conjunction with equations 30 and 31 requires knowledge of ε_{hf}, δ_h (or K_{hf}), and ε_{ht}. Note, however, that one has a choice of utilizing an equivalent boundary condition at the bottom of the film by combining the right-hand sides of equations 29 and 31. This approach is apparently a simplification in that it eliminates ε_{hf} and δ_h (or K_{hf}) from the direct statement of the boundary condition. This simplification can be deceptive, though, since it would still be necessary to determine both T_σ for the right-hand side of equation 29 and T_f for the temperature distribution below the film, and there is apparently no way to relate T_σ and T_f except by considering the transport through the film.

Miller and others (1975) summarized several empirical expressions which have been developed for K_{hf} or equivalent parameters. Most of these expressions related K_{hf} to wind-generated mixing and therefore would apply to large and(or) slowly flowing bodies of water where the primary mixing is wind generated. Nevertheless, there are significant differences between the values of K_{hf} which would be prdicted by the various expressions for a given set of conditions. Furthermore, there is little information available on K_{hf} for situations where a significant part of the turbulence in the surface region is generated by the flow itself. As Holley (1973) noted, the mechanics of heat transfer in the film are closely related to the mass transfer during reaeration. Therefore, a proper use of the analogy between heat and mass transfer in the film region may prove useful in gaining better definition of the transfer coefficients for both processes.

HEAT TRANSFER MECHANISMS AT CHANNEL BOTTOM

Figure 6 depicts the various mechanisms of heat transfer at the bottom boundary or the bed. The arrows in the figure indicate the positive directions used in defining the various ϕ' terms. The prime on ϕ and $d\lambda$ and the subscript β on other terms are used to refer to values at the bed. Other symbols were defined in the discussion of the kinematical boundary conditions at the bed. The heat flux associated with the various mechanisms will be identified as follows:

Symbol	Heat flux associated with
ϕ'_I	= heat content of infiltrated water
ϕ'_s	= advection and diffusion of sediment
ϕ'_w	= advection and diffusion of water
ϕ'_{si}	= incident solar radiation at the bed
ϕ'_{sr}	= reflected solar radiation at the bed
ϕ'_r	= net solar radiation input at the bed = $(\phi'_{si} - \phi'_{sr})$
ϕ'_c	= conduction in bed
ϕ'_d	= diffusion of heat above bed

The time rate of change of heat stored in the incremental volume, $d\lambda'$, per unit area will be labelled ϕ'_λ. The terms ϕ'_{si}, ϕ'_{sr}, and ϕ'_λ are defined per unit horizontal area, while the remainder of the ϕ' terms are defined per unit of bed area. The incident radiation at the bed, ϕ'_{si}, is the part of the transmitted radiation, ϕ_{st}, which is not absorbed by the suspension between the free surface and the bed.

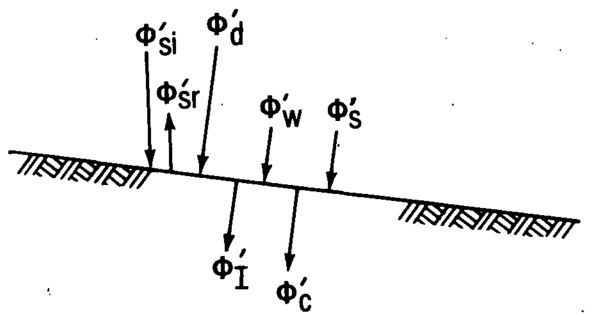

Figure 6.--Heat fluxes at channel bed.

Mathematically, the interface between the suspension and the bed is defined as a smooth surface, B, as depicted in figure 1. Physically, however, the interface region consists of both grains and pores. The part of ϕ'_r which passes through the pore space in the first "layer" of grains will be absorbed in a lower "layer", so that essentially all of ϕ'_r will be absorbed within a depth of approximately two grain diameters at the moving interface. Thus, it will be assumed that ϕ'_r is completely absorbed by the bed material at the mathematically defined interface. Then ϕ'_r effectively represents the rate of heat input for a heat source term at the moving interface and must be considered in writing the thermal boundary condition there.

The terms ϕ'_I, ϕ'_s, ϕ'_w, and ϕ'_λ represent the movement or storage of heat contained in water, sediment, or suspension, and can be expressed by analogy to previous similar terms at the water surface:

$$\phi'_I = I\rho_w C_{pw}(T_\beta - T_r) \qquad (33)$$

$$\phi'_s = (\vec{n}\cdot\vec{v}_s c_s + \vec{n}\cdot\vec{D}_s c_s)_\beta C_{ps}(T_\beta - T_r) \qquad (34)$$

$$\phi'_w = (\vec{n}\cdot\vec{V}_w c_w + \vec{n}\cdot\vec{D}_w c_w)_\beta C_{pw}(T_\beta - T_r) \qquad (35)$$

$$\phi_\lambda' = \frac{\partial B}{\partial t}\left[C_{pw}\rho_w\theta + C_{ps}\rho_s\ (1-\theta) - C_p\rho\right]_\beta (T_\beta - T_r) \qquad (36)$$

It is assumed that the temperature distribution is continuous at the bed, so that the suspension, the bed material, and the infiltration (either positive or negative) all have the same temperature, T_β, at the bed. This condition is part of the statement of the thermal boundary conditions at the bed. Under some circumstances T_β may be significantly different from the temperature of water away from the bed. The diffusive flux, ϕ_d' will be expressed as

$$\phi_d' = (\vec{n}\cdot\vec{D}_T T)_\beta\ \rho C_p \qquad (37)$$

Note again that this term, like equation 25, comes from a temperature gradient at the bed and is not related to ϕ_s' and ϕ_w', which are related to c_s and c_w.

THERMAL BOUNDARY CONDITION AT CHANNEL BOTTOM

The thermal boundary condition at the channel bed can be derived by the law of heat conservation, considering all fluxes illustrated in figures 3 and 6. The complete form of the boundary condition at the bed is given by

$$\phi_r'\ dA - \phi_I' d\beta + (\phi_s' + \phi_w')d\beta - \phi_c' d\beta + \phi_d' d\beta = \phi_\lambda' dA \qquad (38)$$

Equation 38 may be simplified by combining it with the kinematical boundary conditions at the bed, namely, equations 15 and 16. First, transform equations 15 and 16 to heat flux equations by multiplying equation 15 by $C_{ps}(T_\beta - T_r)$ and equation 16 by $C_{pw}(T_\beta - T_r)$. Then combine the two resulting equations and use the definition in equations 33 through 36 to show that

$$-\phi_I' d\beta + (\phi_s' + \phi_w')d\beta = \phi_\lambda' dA \qquad (39)$$

Subtracting equation 39 from equation 38, the reduced thermal boundary condition at the bed is

$$\phi_d' = -\phi_r' \frac{dA}{d\beta} + \phi_c' \qquad (40)$$

Equation 40 may be interpreted as meaning that the radiation absorbed at the bed must be balanced by diffusion in the suspension and by conduction in the bed material, even when the bed is moving vertically due to entrainment or deposition.

Equations 38 and 40 both represent the same boundary condition, but equation 38 is normally the most convenient form when the three-dimensional thermal energy equation is being depth-averaged. In the depth-averaging process, ϕ_r', ϕ_s', ϕ_w', ϕ_d', and ϕ_λ' are eliminated, while ϕ_I' and ϕ_c' remain as source and sink terms of the depth-averaged equation. When solving for vertically nonuniform distributions of temperature, however, equation 40 is normally convenient to use.

Use of equations 38 and 40 is similar to that of equations 26 and 28, as explained previously. As before, ϕ_d' can be expressed in terms of a scalar diffusion coefficient, $\varepsilon_{h\beta}$, normal to the bed, so that

$$\varepsilon_{h\beta}\rho C_p\left(\frac{\partial T}{\partial n}\right)_\beta = \phi_r' \frac{dA}{d\beta} - \phi_c' \qquad (41)$$

Care should be taken to assure that any variation of ε_h near the bed is correctly represented in evaluating $\varepsilon_{h\beta}$.

The above derivation included the incident solar radiation at the bed. In some cases, this can apparently be a significant heat transfer mechanism. Brown (1972) observed that a substantial amount of shortwave radiation was absorbed directly by the solid rock bed in extremely shallow streams. Bowles and others (1976) and Comer and others (1975) came to a similar conclusion. Information on the attenuation of shortwave radiation with depth (for example, Dake and Harleman, 1969) can be used to determine the vertical distribution of absorption of radiative energy in the water and to determine the potential for significant amounts of radiation reaching the bed. If there is significant incident radiation at the bed, care must be taken to assure that the evaluation of the interface temperature, T_β, adequately reflects such a condition. In particular, it must be recognized that, although ϕ_r' may be a continuous heat source at the bed, the heat is being absorbed by different grains at different times during deposition or entrainment. Comer and others (1975) presented an analysis and some measurements of the temperature distribution in a vertical soil column beneath a stationary stream bed in connection with analyzing the temperature variations in the stream. They gave an expression equivalent to equation 40 as the boundary condition for the nonmovable bed case which they were studying and discussed the determination of T_β for that case.

Equation 41 can be used as the thermal boundary condition for analyzing vertically nonuniform temperature distributions without considerations of the magnitude of infiltration $(I>0)$ or accretion from ground water $(I<0)$. In contrast to the free surface situation, where mass fluxes due to precipitation and evaporation are normally small, the magnitude of mass flux, I, at the bed could be substantial and could lead to substantial cooling or heating of river water (Comer and others, 1975; Jackman and Yotsukura, 1977).

CONCLUSIONS

General thermal boundary conditions have been derived for both the free surface and the channel bottom for unsteady flow with moving boundaries. Since any movement of the bottom boundary normally involves sediment movement, the presence of suspended sediment was considered in the derivation of boundary conditions. Kinematical boundary conditions for water, sediment, and the suspension have also been presented, since they can be used to simplify the thermal boundary conditions.

Equations 26 and 38 are complete forms of the thermal boundary conditions at the surface and the bed, respectively. These forms normally are used in depth-averaging of the thermal-energy conservation equation and provide source and sink terms in the resulting equation.

In solving the three-dimensional thermal energy equation for vertically nonuniform temperature distribution, however, reduced forms of thermal boundary conditions, namely, equations 28 and 40, can be used as the conditions to be satisfied at the free surface and at the channel bottom, respectively. Both equations are independent of water and sediment fluxes across the boundaries except for a small part due to precipitation. Equation 28 for the free surface is of the Cauchy type in that it gives a relation between the temperature and the normal derivative of the temperature at the surface. Use of equation 28 requires the knowledge of surface film heat transfer characteristics, such as the heat transfer coefficient for the film. Equation 40 for the bed states that there must be a balance among the diffusive thermal fluxes on the two sides of the boundary and the net radiation input at the bed.

SELECTED REFERENCES

Anderson, E. R., 1954, Energy-budget studies, water-loss investigations, Lake Hefner studies, Technical report: U.S. Geol. Survey Prof. Paper 269, p. 71-119.

Bird, R. B., Stewart, W. E., and Lightfoot, E. N., 1966, Transport phenomena: John Wiley & Sons, Inc., New York, N.Y., p. 496-501.

Bowles, D. S., Comer, L. E., and Grenney, W. J., 1976, Dynamic stream temperature model for unsteady flow: Rept. PRWG 156, Utah Water Research Laboratory, Utah State University, Logan, Utah, 148 p.

Brown, G. W., 1972, An improved temperature prediction model for small streams: Water Resources Research Inst. Rept. WRRI-16, Oregon State Univ., Corvallis, Oreg., 20 p.

Chen, C. L., 1971, Rate of thermally polluted surface water in rivers: Am. Soc. of Civil Engineers, Jour. of Sanitary Eng. Div., v. 97, no. SA3, p. 311-331.

Comer, L. E., Bowles, D. S., and Grenney, W. J., 1975, Field investigation and mathematical model for heat transfer process in bed of small stream: Utah Water Research Laboratory, Utah State Univ., Logan, Utah, Final Rept. prepared for Office of Hydrology, NOAA, 103 p.

Dake, J. M. K., and Harleman, D. R. F., 1969, Thermal stratification in lakes: Analytical and laboratory studies: Water Resources Research, v. 5, no. 2, p. 484-495.

Edinger, J. E., and Geyer, J. C., 1965, Heat exchange in the environment: EEI Pub. no. 65-902, Edison Electric Inst., New York, N.Y., 259 p.

Ewing, G., and McAlister, E. D., 1960, On the thermal boundary layer of the ocean: Science, v. 131, no. 1374.

Harleman, D. R. F., 1972, Longitudinal temperature distribution in rivers and estuaries, one-dimensional mathematical models: Engineering Aspects of Heat Disposal from Power Generator, MIT Summer Session, Cambridge, Mass.

Holley, E. R., 1973, Diffusion and boundary layer concepts in aeration through liquid surfaces: Water Research, v. 7, p. 559-573.

Jackman, A. P., and Yotsukura, N., 1977, Thermal loading of natural streams: U.S. Geol. Survey Prof. Paper 991, 39 p.

Jobson, H. E., and Yotsukura, N., 1973, Mechanics of heat transfer in non-stratified open-channel flows: Environmental Impact on Rivers edited by H. W. Shen, Fort Collins, Colo. p. 8.1-8.67.

Miller, A. W., Street, R. L., and Hsu, E. Y., 1975, The structure of the aqueous thermal-sublayer at an air-water interface: Rept. 195, Dept. of Civil Engineering, Stanford Univ., Stanford, Calif., 193 p.

Morse, W. L., 1970, Stream temperature prediction model: Water Resources Research, v. 6, no. 1, p. 290-302.

Paily, P. P., Macagno, E. O., and Kennedy, J. F., 1974, Winter-regime thermal response of heated streams: Am. Soc. of Civil Engineers, Jour. of Hydraulics Div., v. 100, no. HY4, p. 531-551.

Prych, E. A., Haushild, W. L., and Stoner, J. D., 1976, Numerical model of the salt-wedge reach of the Duwamish River estuary, King County, Washington: U.S. Geol. Survey Prof. Paper 990, 34 p.

Ryan, P. J., and Harleman, D. R. F., 1973, An analytical and experimental study of transient cooling pond behavior: Rept. 161, Parsons Lab. for Water Resources and Hydrodynamics, MIT, Cambridge, Mass., 439 p.

Schiller, E. J., and Sayre, W. W., 1975, Vertical temperature profiles in open-channel flow: Am. Soc. of Civil Engineers, Jour. of Hydraulics Div., v. 101, no. HY6, p. 749-761.

Sokolnikoff, I. S. and Redheffer, R. M., 1958, Mathematics of physics and modern engineering: McGraw-Hill Book Co. Inc., New York, N.Y., 812 p.

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