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Theoretical Analysis of Forced Convective Heat Transfer in Regional Ground-Water Flow

ABSTRACT

Among the numerous processes that will cause anomalous temperature distributions in geologic basins is the spatial redistribution of heat by moving ground water. This problem is examined by solving the energy equation for the simultaneous transport of water by hydraulic gradients and heat by forced convection. The factors that affect the temperature distribution in a given basin include the intrinsic properties of the medium and contained fluid—namely, the thermal diffusivity of the solid-fluid complex and the hydraulic conductivity, the water-table configuration, and the ratio of basin depth to basin length. The severity of an anomalous geothermal gradient or temperature measurement depends primarily on the relative magnitude of the ratio of hydraulic conductivity to thermal diffusivity, and on the geometry of the flow field. A dimensionless group may be formulated from these parameters, and provides a relative measure of the simultaneous transport of heat by the bulk motion of the fluid to that by pure conduction. Solutions to the equation itself indicate that convective heat losses in ground-water recharge areas are balanced by convective heat gains in discharge areas. The geothermal gradient accordingly increases with increasing depth in recharge areas, decreases with increasing depth in discharge areas, and is a manifestation of pure conduction at the hinge line separating areas of recharge and discharge.

INTRODUCTION

Anomalous temperature gradients in subsurface formations have been attributed to the following mechanisms: (1) differences in the thermal conductivity of the formations, (2) geologically recent intrusions of magma or other sources of heat production, and (3) the spatial redistribution of heat by moving ground water. The last category has been the subject of numerous descriptive studies by geophysicists and hydrogeologists. Schneider (1964, p. 214) observed a very small temperature gradient in recharge areas in carbonate rocks and suggested that the influx of meteoric water was sufficient to redistribute heat derived from the Earth's interior. On the basis of extensive field work in a large carbonate region in Nevada, Mifflin (1968, p. 31) suggested that the temperature of thermal springs is related in an approximate manner to the depth of water circulation. An early study by Van Orstrand (1934, p. 996) focused on convective alterations of the geothermal gradient as observed in the permeable Wall Creek Sands of the Salt Creek oil field, Wyoming.

The primary motivation of this study is to complement these and similar field studies with a theoretical approach to the simultaneous transfer of heat and water in regional ground-water flow. In particular, emphasis will be focused on both approximate and exact solutions that provide insight into the various temperature distributions which might occur in regional porous bodies; emphasis will also be made on examination of the important limiting cases.

HEAT TRANSFER IN REGIONAL GROUND-WATER FLOW

Transport processes can be broadly categorized according to their occurrence in stationary or moving media. Examples of transport in stationary media include heat conduction in solids and chemical diffusion through a membrane or stagnant gas. When the medium is subject to either forced or natural motion, the transport process is said to take place in a

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moving medium. Heat (or mass) transfer associated with the movement of a fluid is termed convective heat (or mass) transfer.

The basic equation which governs convective heat transfer has been discussed in several texts on transport phenomena and has been derived for porous media by Stallman (1963) and Slattery (1972, p. 402–405). The governing equation is a simplified version of the equation of energy (Bird and others, 1960, p. 316), and is expressed

$$\nabla\left(\frac{\kappa}{\rho C}\nabla T\right) - (\mathbf{v}\cdot\nabla T + T\nabla\cdot\mathbf{v}) + q = \frac{\partial T}{\partial t} \tag{1}$$

where κ is the thermal conductivity of the fluid-solid complex, ρ is the density and C is the specific heat of the fluid, T is temperature, **v** is a vector designating the velocity of ground water, q is a source term, and t is time. In the absence of a moving medium (**v** = 0) and heat sources, equation 1 reduces to the well-known heat conduction (or diffusion) equation.

In equation 1, the ground-water flow is described by the velocity field v (x, y, z). Two additional equations, referred to as equations of change, are required to determine this field. The first of these, the continuity equation, is expressed

$$\operatorname{div}(\rho v) + \partial(\rho n)/\partial t = 0 \tag{2}$$

where n is porosity, and all other terms are as previously defined. The second of the required equations is an equation of motion, which is specified in accordance with the type of motion that is prevalent. Two limiting types are possible—forced and free.

With forced convection, such as ground-water movement in the absence of density gradients, fluid motion is due to externally imposed forces. With this type of motion in shallow subsurface formations, the equation of motion is merely the gradient of some scalar potential—that is, Darcy's law, expressed as

$$v = -\frac{K}{g}\nabla\Phi \tag{3}$$

where K is the hydraulic conductivity, g is the acceleration due to gravity, and Φ is the hydraulic potential.

Free (or natural) convection is probably the dominant type of fluid motion in hydrothermal systems where the bulk of the liquid discharge is in the form of steam or hot water. In free convection, the motion of the fluid is due largely to density variations which are caused by temperature gradients. Hence, the equations of motion and energy are coupled, in that temperature terms appear in both. It follows, of course, that there no longer exists a scalar potential of which the gradient is the velocity, so that Darcy's law no longer describes the motion. Free convection is not within the scope of this paper.

It is emphasized that forced and free convection represent two limiting conditions. In the case of the former, the buoyancy forces are assumed to be negligible; in the case of the latter, fluid motion is expressed entirely in terms of buoyancy.

Conceptualization of Regional Flow

Given the problem of forced convection, it is now necessary to supply some specific information regarding the moving medium. A useful conceptualization of regional ground-water flow for analytical purposes incorporates the following assumptions (Toth, 1963, p. 4797): (1) The porous medium is isotropic and homogeneous to a specified depth, below which there exists a horizontal impermeable basement. (2) Flow is steady and is restricted to a two-dimensional vertical section. The topography can be approximated by simple curves, and the water table is a subdued replica of the topography. (3) The upper boundary of the flow field is the water table, the lower boundary is the impermeable basement, and the lateral boundaries are major ground-water divides.

Statement 3 is shown schematically in Figures 1a and 1b. The potential referred to in the figure is the hydraulic potential, described by Hubbert (1940, p. 802) as

$$\Phi = gz + \int_{p_a}^{p}\frac{dp}{\rho} \tag{4}$$

where Φ is the hydraulic potential at any point in the field, z is the elevation above a standard datum; p is the pressure at any point, and p_a is atmospheric pressure. Recognizing that the total head h is $z + p/\rho g$, equation 4 reduces to $\Phi = gh$.

The energy equation can now be re-expressed by incorporating Darcy's law and the assumption of steady flow. Further assumptions include steady-state temperature conditions, constant thermal conductivity, and an absence of heat sources:

$$\alpha \nabla^2 T - \left(-\frac{K}{g} \nabla \Phi \cdot \nabla T \right) = 0 \qquad (5)$$

where α is a mixed thermal diffusivity, or thermal conductivity of the solid-fluid complex divided by the product of the density and specific heat of the fluid. The convective term in equation 5 is the dot product of two vectors, $\nabla\Phi$ and ∇T, and merely states that the velocity vector of ground-water flow along the temperature gradient redistributes heat by convection. For the two-dimensional problem under consideration, the velocity term $\frac{K}{g}\nabla\Phi$ is not constant spatially, and it is first necessary to determine the potential distribution $\Phi(x, z)$ in the region of interest in order that $\nabla\Phi(x, z)$ can be determined. The potential distribution can be found by solving Laplace's equation, which is easily obtained by combining the continuity equation with Darcy's law:

$$\nabla \cdot v = \nabla \left(-\frac{K}{g} \nabla \Phi \right) = 0 \,. \qquad (6)$$

A solution to Laplace's equation for the appropriate boundary conditions will yield the potential distribution $\Phi(x, z)$ of the ground-water flow region. The gradient of $\Phi(x, z)$ is the quantity required in the energy equation. Hence, the forced convection problem is solved by first solving the flow problem independently of the temperature field and then incorporating the results in the energy equation.

FLOW PROBLEM

The schematic diagram in Figure 1a represents a homogeneous, isotropic region in which Laplace's equation is to be solved with appropriate boundary conditions. As there is no lateral flow across major topographic highs and lows, the two vertical boundaries are rendered "impermeable" by setting the gradient across them equal to zero—that is to say, $\partial\Phi/\partial x = 0$. The same procedure applies to the impermeable bottom, where z is the direction normal to the boundary.

A fourth condition is required along the upper surface of flow, or $z = z_0$. The hydraulic

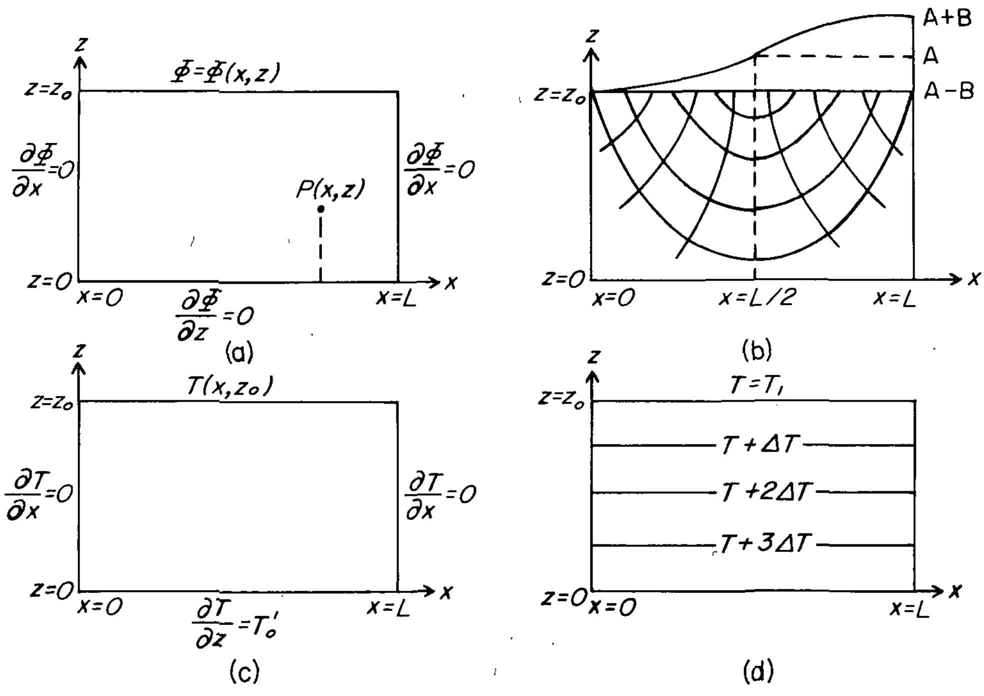

Figure 1. Two-dimensional diagrams showing (a) the boundary conditions for the flow problem, (b) the flow field for the water-table configuration $A - B \cos(\pi x/L)$, (c) the boundary conditions for the steady-state temperature distribution, and (d) the temperature field of pure conduction for a constant temperature upper boundary.

head at any point on this surface is taken as the elevation of the water table above the point in question. The values of z_0 (and therefore Φ) along the water table are a function of the x coordinate; hence, the boundary condition at $z = z_0$ is designated $\Phi = \Phi(x, z_0)$, where $\Phi(x, z_0)$ is the equation of the water table. Arguments supporting this assumption as well as others that have been made are discussed by Toth (1963, p. 4796–4797).

The task is now to provide a reasonably general solution to the flow problem discussed above. By "general," we mean a solution for Laplace's equation in the region of Figure 1a where the upper boundary is not specified. The solution is determined from a separation of variables technique, and is

$$\Phi(x, z) = a_0 + \sum_{n=1}^{\infty} a_n \cosh \frac{n\pi z}{L} \cos \frac{n\pi x}{L} \qquad (7)$$

where the coefficients a_0 and a_n are determined from the equation of the water table for various special cases. The coefficients are determined from the following equations:

$$a_0 = \frac{1}{L} \int_0^L \Phi(x, z_0) dx \qquad (8)$$

$$a_n = \frac{2}{L \cosh (n\pi z_0/L)} \int_0^L \Phi(x, z_0) \cos \frac{n\pi x}{L} dx .$$

As an example of a special case that will be used later in a heat-transfer problem, assume that the equation of the water table $\Phi(x, z_0)$ equals $A - B \cos \frac{\pi x}{L}$ (Fig. 1b). The coefficents of the general solution are

$$a_0 = A; a_1 = \frac{-B}{\cosh (\pi z_0/L)}; \text{other } a\text{'s} = 0 .$$

Equation 7 then becomes

$$\Phi(x, z) = A - \left[\frac{B \cosh (\pi z/L)}{\cosh (\pi z_0/L)}\right] \cos (\pi x/L) . \qquad (9)$$

This case is interesting because the simplicity of the solution permits a good understanding of the geometrical controls on the spatial distribution of the potential. From equation 9 and Figure 1b, it is evident that at the points $z = z_0$ for $x = 0$, $x = L$, and $x = L/2$, the value of the potential $\Phi(x, z_0)$ is $A - B$, $A + B$, and A, respectively. Furthermore, at $x = L/2$ for all z, $\Phi(x, z) = A$; that is, a vertical equipotential line exists at x equals L/2. For $L/2 < x \leq L$ and all z, $\Phi(x, z)$ is greater than A, and for $0 < x < L/2$ for all z, $\Phi(x, z)$ is less than A. The vertical equipotential line at $x = L/2$ then corresponds to a midline of flow (Toth, 1962, p. 4382), wherein flow lines up gradient from the midline are directed downward with respect to the water table, and flow lines down gradient from the midline are directed upward with respect to the water table. The former region is referred to as a recharge area, and the latter region is referred to as a discharge area. This is illustrated in Figure 1b, where a flow net has been sketched from a few calculations of the potential at various points.

Before leaving the flow problem, it is important to note the changes in the flow field as the ratio z_0:L changes. As this ratio becomes small, ($\ll 1$), the lateral extent of the flow field becomes greater than the depth of active circulation, and the so-called midline becomes an extensive area of lateral flow characterized by numerous near-vertical equipotential lines. In essence, large changes in potential in a vertical direction occur only near the lateral boundaries of the flow region. As this ratio becomes large ($\gg 1$), the region of lateral flow vanishes, or shrinks to a line, with major components of flow being directed both upward and downward. In essence, large changes in potential in a vertical direction occur throughout the flow region. As ground-water movement is the major factor contributing to the spatial redistribution of heat derived from the Earth's interior, the geometry of the flow field is suspected of being of first-rank importance insofar as it contributes to this redistribution. This will be demonstrated below.

APPROXIMATE SOLUTION TO THE CONVECTION EQUATION

A first-order iteration of the convection equation proceeds by assuming that convection has a small effect in redistributing heat so that the temperature distribution in the subsurface is due largely to pure conduction. Thus, we solve the pure conduction problem of Figure 1c to obtain T(x, z) and then substitute the gradient of this expression in the convective term $(K/g)\nabla\Phi \cdot \nabla T$ of equation 5. This is a common procedure of perturbing the system to allow the convection effects to enter. As explained above, $\nabla\Phi$ is determined independently from the flow problem.

A steady-state solution to the conduction problem of Figure 1c where the upper boundary is not specified and T_0' is a constant temperature gradient across the lower boundary is

$$T(x, z) = b_0 + T_0'z + \sum_{n=1}^{\infty} b_n \cosh(\pi n z/L) \cos(\pi n x/L) \qquad (10)$$

where the coefficients are found from the upper boundary condition. If the upper boundary is taken as a constant T_1, equation 10 becomes

$$T(x, z) = T_1 + T_0'(z - z_0) \,. \qquad (11)$$

The isotherms for this simple system are sketched in Figure 1d.

The forced convection problem of equation 5 is now solved by making appropriate use of the general flow solution described by equation 7 and the pure conduction solution as given in equation 11:

$$T(x, z)) = T_1 + T_0'(z - z_0) + T_0' K/2\alpha \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) \times \left[(z_0 - z) \cosh(n\pi z/L) + \frac{L \sinh[n\pi(z - z_0)/L]}{n\pi \cosh(n\pi z_0/L)} \right]. \qquad (12)$$

Equation 12 is a first-order iteration in powers of K/α of the temperature distribution in a region characterized by the boundary conditions of Figure 1. Again, the a_n are found from the water-table configuration of the special cases. In this reasonably general form, the temperature distribution as described by equation 12 is controlled by the following essential features:

1. The first few terms $T_1 + T_0'(z - z_0)$ describe the temperature distribution for the case pure conduction only (see equation 11). Thus, for the case of no flow ($a_n = 0$), or very high thermal diffusivity ($\alpha \rightarrow \infty$), the heat-transfer system is dominated by pure conduction. When convective transport is operative (both a_n and $K \neq 0$), the flow field accordingly modifies the temperature distribution of pure conduction.

2. The second interesting component of equation 12 contains the quantity K/α, which is the ratio of hydraulic conductivity to thermal diffusivity. The expression K/α has the physical dimensions length^{-1}, and may be taken as a measure of the relative efficiency of an element of porous material for the simultaneous transport of fluid by hydraulic gradients and heat by conduction. If this ratio is multiplied by some length which could be considered characteristic of fluid motion in a regional porous body, a numeric or dimensionless quantity would result. It will be seen shortly that this characteristic length is provided by an evaluation of the coefficients a_n when a special case water-table configuration is specified. Thus, once a specific-flow geometry is identified, there will result a dimensionless quantity $K\ell/\alpha$, where ℓ is a characteristic length designating some critical dimension of the water table. Such numerics occur frequently in transport problems, and here may be taken as a measure of the relative efficiency of a regional porous body for the simultaneous transport of two properties, water by hydraulic gradients and heat by pure conduction. The quantity $K\ell/\alpha$ will be large in regional systems which conduct fluid more effectively than heat, and small in regional systems which conduct heat more effectively than fluid. As forced convection is directly dependent on the bulk motion of the ambient ground water, the ratio may also be taken as a measure of the relative efficiency for the simultaneous transport of heat by forced convection and pure conduction. This latter statement, however, is not exactly true in that the influence of basin length and depth on the flow path must also be considered. It follows that for the assumptions made in the first-order iteration, the distribution $T(x, z)$ as given in equation 12 becomes more exact as the ratio $K\ell/\alpha$ becomes small.

3. The third interesting component of equation 12 is the terms within the summation sign. Clearly, these trigonometric and hyperbolic functions deal almost exclusively with the geometry of the porous medium, in particular the ratio of basin depth z_0 to length L. If z_0 is made to approach zero so that z_0/L is small, the flow paths will be largely horizontal, and the cosine hyperbolic terms take on their smallest values, while the sine hyperbolic term goes to zero. This is the case in which there is the least convective alteration of the temperature distribution produced by pure conduction. If z_0 is made to approach infinity so that z_0/L is large, the flow paths differ markedly

from the horizontal, and both sine and cosine hyperbolic terms take on their maximum values. This is the case of maximum convective alteration of the temperature distribution caused by pure conduction. The mathematics here illustrates, with reference to the convective term in equation 5, that convective heat transport is eliminated when the hydraulic gradient of fluid flow and the temperature gradient of pure conduction are normal to each other; that is, the streamlines of fluid flow are collinear with the isotherms of heat conduction. The maximum convective effects are achieved when the respective gradients are collinear.

Heat Flow at the Surface and the Geothermal Gradient

An expression of the geothermal gradient that gives some information on heat flow at the surface may be obtained by taking the derivative of equation 12 with respect to z and evaluating the expression at z equals z_0. Performing these operations on equation 12 gives

$$\left.\frac{\partial T}{\partial z}\right|_{z=z_0} = T_0'/2 \times \left[2 - \frac{K}{\alpha}\sum_{n=1}^{\infty} a_n \cos(\pi n x/L) \cdot \sinh^2(\pi n z_0/L)/\cosh(\pi n z_0/L)\right]. \quad (13)$$

In this form, equation 13 gives the temperature gradient in the z direction evaluated at the upper surface $z = z_0$, with $\Phi(x, z)$ being arbitrary temporarily. Note that in the absence of convection, the gradient at all x equals T_0', or the gradient permitted across the lower boundary of flow. This, of course, is to be expected for steady-state heat conduction in a homogeneous, isotropic region in the absence of both convection and heat sources within the region. For a_n and K not equal to zero, the gradient at the surface can be greater than T_0', or less than T_0', depending on the net sign of the convective terms.

Further examination of the manner in which fluid motion redistributes the heat of conduction from the Earth's interior requires a special case flow situation. For ease in interpretation, the potential distribution $\Phi(x, z)$ of equation 9 is used to describe the flow field. Equation 13 then becomes

$$\left.\frac{\partial T}{\partial z}\right|_{z=z_0} = T_0'/2\left[2 + \frac{KB}{\alpha}\cos(\pi x/L) \cdot \tanh^2(\pi z_0/L)\right] \quad (14)$$

where B is the mean water-table elevation above the surface z_0 (Fig. 1b). The following relations are easily determined: at $x = 0$, $\partial T/\partial z > T_0'$; at $x = L/2$, $\partial T/\partial z = T_0'$; and, at $x = L$, $\partial T/\partial z < T_0'$. Hence, the heat flux at the surface is greater in discharge areas than in recharge areas. At the midline of the flow region ($x = L/2$), the gradient at the surface is the same as in pure conduction. The reasons for this have already been stated with regard to the vector orientation of two properties.

Influence of the Flow Geometry on the Temperature Field

The few calculations above suggest that the average geothermal gradient as measured through the total thickness of a flow region is greater in ground-water discharge areas than in recharge areas. The nature of the flow as distributed on both sides of the theoretical midline suggests further that the temperature increases with increasing depth more rapidly in discharge areas than it does in recharge areas. At the midline, or within an area of lateral flow, the temperature distribution is a manifestation of pure conduction and is unaffected by convective transport. Although these statements are true in general, the geometry of the flow pattern will determine the extent to which they are significant. To illustrate this point, the first-order approximation of equation 12 is solved for the simple flow solution described by equation 9:

$$T(x, z) = T_1 + T_0'(z - z_0) - \frac{T_0' KB}{2\alpha} \times \frac{\cos(\pi x/L)}{\cosh(\pi z_0/L)} \times \left\{(z_0 - z)\cosh(\pi z/L) + \frac{L}{\pi}\left[\sinh\frac{\pi(z - z_0)}{L} \Big/ \cosh(\pi z_0/L)\right]\right\}. \quad (15)$$

Equation 15 is evaluated for the ratio z_0/L

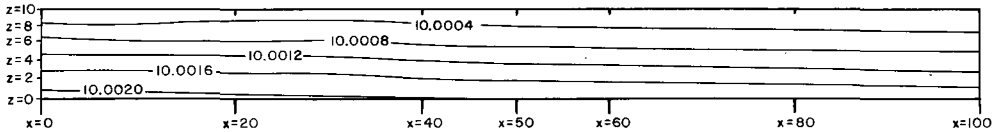

Figure 2. Distribution of isotherms for a flow field with a basin depth to length ratio of 0.1.

equals 0.1 and 1.0, and the results plotted in Figures 2 and 3, respectively. Other pertinent parameters include $K = 1 \cdot 10^{-2}$ cm/sec; $\alpha = 1 \cdot 10^{-2}$ cm^{-2}/sec; $T_1 = 10°C$ (assumed to represent mean air temperature); and T_0' is taken as a representative gradient across the lower surface, $2 \cdot 10^{-2}$°C/cm. The value for B is taken as equal to cosh $(\pi z_0/L)$, so that for both cases, the quantity $T_0'KB/2\alpha \cosh(\pi z_0/L)$ equals $1 \cdot 10^{-4}$, and has the dimensions of a temperature gradient. This means that the differences in temperature distribution depicted in Figures 2 and 3 are the result of differences in flow geometry.

These figures corroborate the general statements given above and demonstrate further that the isotherms approach the case of pure conduction when basin length greatly exceeds basin depth (Fig. 2), and that they deviate markedly from this case whenever the depth of active circulation is increased relative to basin length (Fig. 3).

Figure 4 shows the temperature distribution versus depth for a few points taken from Figures 2 and 3.

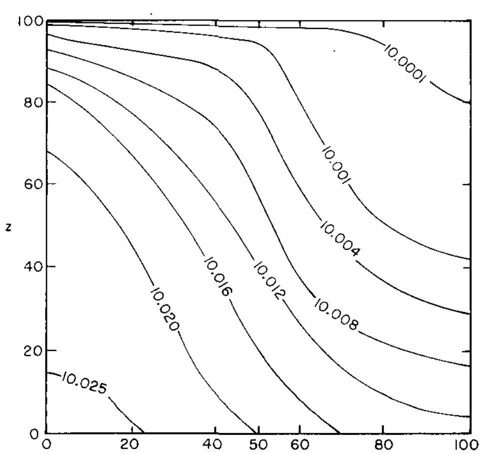

Figure 3. Distribution of isotherms for a flow field with a basin depth to length ratio of 1.0.

Dimensionless Group KB/α

The dimensionless quantity KB/α appears in the pertinent equations 14 and 15, where B is a characteristic length designating the mean water-table elevation above z_0. The quantity B is obviously a measure of the forcing function contributing to convective transport in regional flow. As mentioned above, this numeric may be taken as a measure of the relative efficiency of a regional porous body for the simultaneous transport of water by hydraulic gradients and heat by pure conduction. This quantity appears to be an analog of the Peclet number, which is a ratio of heat transfer by bulk motion of a fluid to its transfer by conduction. The Peclet number occurs frequently in forced convection problems; it is easily obtained from the product of the Reynolds and Prandtl numbers

$$N_{Pe} = N_{Re} \cdot N_{Pr} = \frac{\ell v \rho}{\mu} \cdot \frac{\mu}{\rho \alpha} = \frac{\ell v}{\alpha} \quad (16)$$

where v is velocity, ℓ is a characteristic length, μ is viscosity, ρ is density, and α is the thermal diffusivity, all for a fluid system. It follows that the group KB/α should likewise be easily derived from the product of the Reynolds and Prandtl numbers for a porous system. Such a derivation is of interest because the Reynolds number is widely known as a nonunique quantity because the characteristic length ℓ and the velocity v can be chosen in many ways. A popular choice of experimentalists in porous flow is an average grain diameter for the critical flow dimension and an apparent velocity from Darcy's law for the velocity term. There are, however, other candidates for ℓ, and these are generally selected on the basis of the scale of the flow apparatus from which measurements are taken.

For the regional scale problem under consideration, the selection of the characteristic length is no longer arbitrary, and it must be taken as some critical dimension of the water table—that is, taken as a measure of the external forces contributing to the forced convection. Further examination of the Reynolds number from the point of view of dimensions of its component parts suggests that the hydraulic conductivity is the counterpart of the velocity term. As hydraulic conductivity is a measure of both the medium and the fluid, it may be conveniently replaced by $k\rho g/\mu$, where g is the acceleration due to gravity and k is the intrinsic permeability, the latter used in reference to a property of the medium only (Hubbert, 1940, p. 819). This facilitates the derivation of an analog of the Peclet number as the product of the Reynolds and Prantdl numbers

$$N_{Re} \cdot N_{Pr} = \frac{B(k\rho g/\mu)}{\mu} \cdot \frac{\mu}{\rho\alpha} = \frac{B(k\rho g/\mu)}{\alpha} \quad (17)$$

which is equivalent to KB/α.

The relative efficiency of the regional systems of Figures 2 and 3 for the simultaneous conductance of water and heat may now be measured with the analog of the Peclet number. This simple calculation implies that the regional system of Figure 2 conducts heat and water equally well, whereas the system of Figure 3 has a capacity for water conduction that is approximately 11 times greater than its capacity for heat conduction. When a regional system conducts heat as effectively as it does water, as shown in Figure 2, there is very little convective heat transport. On the other hand, a high capacity for the conductance of water relative to heat gives rise to a high capacity for temperature redistributions by bulk motion of the fluid. As mentioned previously, this last statement is not exactly true in that the geometrical controls on the flow pattern are not incorporated in the dimensionless group. A modified version of the Peclet number which remedies this omission may therefore be useful in studies where only some approximate measure of the importance of forced convection is required. This modified version is arrived at by dimensional arguments, and is

$$\frac{BKz_0/L}{\alpha}. \quad (18)$$

This new group may now be given an unqualified interpretation as a relative measure of heat transfer by bulk motion of a fluid to conductive heat transfer in a regional porous body.

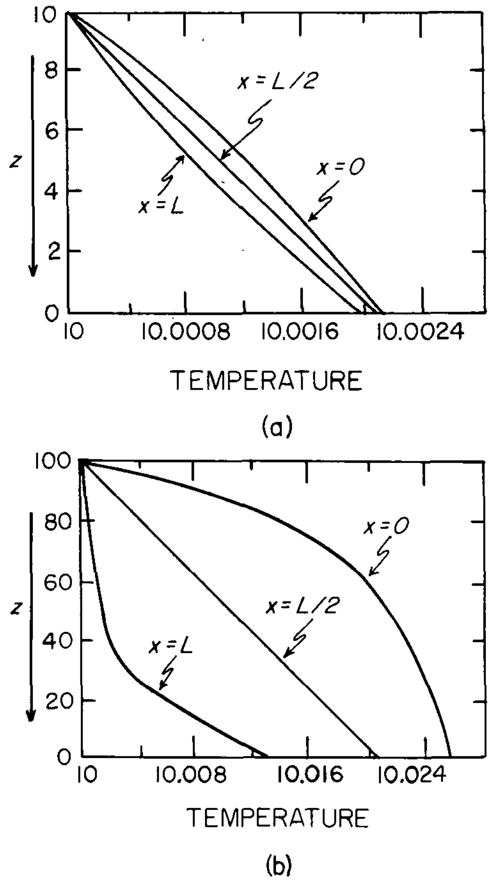

Figure 4. Temperature versus depth at the points x equals O, L/2, and L for (a) a basin with a depth to length ratio of 0.1, and (b) a basin with a depth to length ratio of 1.0.

The number equals 0.1 for the system shown on Figure 2, and approximately 11 for the system shown on Figure 3. As a first guess, it is likely that regional flow systems with a modified Peclet number less than one are characterized by temperature distributions that are not far disturbed from the limiting case of pure conduction. Numbers greater than one appear to be indicative of systems with significant temperature redistributions caused by the bulk motion of the ambient ground water.

GENERALIZING THE RESULTS

The limitations to the generality of the results thus far given include: (1) The only flow field subject to serious consideration is bounded by the simple water-table configuration shown on Figure 1b, and (2) the first-order perturbation solution loses its quantitative correctness when convection is large relative to conduction. The purpose of this section is to demonstrate by methods that are more heuristic than rigorous that the qualitative statements concerning the over-all structure of the solution and the dimensionless group in particular remain valid for: (1) more complex water-table configurations, and (2) any arbitrary ratio of convection to conduction.

Superposition of Flows

A reasonably general water-table configuration consists of a regional slope with superposed local relief so that local flow systems are superposed on a regional flow system (Toth, 1963, p. 4797). The upper boundary is expressed

$$\Phi(x, z_0) = g[z_0 + B'(x/L) + b \sin 2\pi x/\lambda] \qquad (19)$$

where B' is the height of the water table above z_0 at the regional ground-water divide, b is the amplitude of oscillations, and λ is the length of the oscillations so that L/λ is the number of local systems (Fig. 5). The second term on the right-hand side of equation 19 accounts for the regional slope, whereas the last term is the local relief. The structure of the coefficients is determined from equation 8:

$$a_n = \frac{2}{L \cosh (n\pi z_0/L)} \int_0^L g[z_0 + B'(x/L) + b\sin 2\pi x/\lambda] \cos \frac{n\pi x}{L}\, dx$$
$$= B'[f(n, \text{etc.})] + b[(f(n, \text{etc.})] \qquad (20)$$

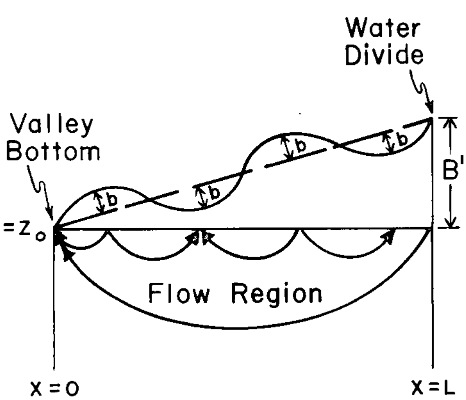

Figure 5. Two-dimensional schematic diagram showing the superposition of local flow on a regional flow system.

where B' and b have units of length and the terms in the brackets are functions of the pertinent variables. The structure of the final result is determined from equation 12:

$$T(x, z) = T_1 + T_0'(z - z_0) - \frac{T_0'KB'}{2\alpha}\{\cdots\} - \frac{T_0'Kb}{2\alpha}\{\cdots\} \qquad (21)$$

where the braces $\{\cdots\}$ correspond to the trigonometric and hyperbolic functions describing the flow patterns due to the corresponding terms in the upper boundary condition. Thus, for each term in the boundary condition, there will be a corresponding term in the final solution with a corresponding number. The terms will be additive, as we are still considering a first-order approximate solution. It follows that if local relief is negligible ($b = 0$), convective heat transport takes place in a regional system bounded by a linear water table whose elevation decreases from the ground-water divide at the topographic high to the major drainage. In this case, B' in the dimensionless number is the characteristic length. If the regional slope is negligible ($B' = 0$), convective transport takes places in the hydrologically isolated local flow systems, with b emerging as the critical dimension of the water table. In the general case, B' will be much larger than b, and the resultant temperature distribution will be dominated by the regional flow system. The midline is once more an important feature, with higher temperatures being encountered in both the regional and local flow systems with distance down gradient from the midline.

An Exact Solution

An exact solution to the convection-conduction problem as specified by the energy equation 5 and the flow problem of equation 9 may be expressed in terms of the periodic Mathieu functions $Ce_r(x)$ and $Se_r(x)$:

$$T(x, z) = T_1 + (e^{-K\Phi(x, z)/2\alpha})f(x, z) \qquad (22)$$

where

$$f(x, z) = \sum_{r=0}^{\infty} h_r[Se_r(i\pi z/L)Ce_r(i\pi z_0/L) - Se_r(i\pi z_0/L)Ce_r(i\pi z/L)]Ce_r(\pi x/L) \qquad (23)$$

where the h_r are coefficients and the i's are imaginary units. The coefficients are determined in order to satisfy the boundary condition dT/dx at z equals zero, equals T_0'. An important feature of this solution is that in the absence of convection, the Mathieu functions become the ordinary trigonometric and hyperbolic functions so that in the limit as convection goes to zero,

$$Ce_r(\pi x/L) \rightarrow \cos(\pi r x/L)$$

$$Ce_r(i\pi z/L) \rightarrow \cosh(\pi r z/L) .$$

Hence, keeping the linear terms of KB/α in equation 22, the approximate result may be recovered.

Further analogies in structure may be examined by rewriting equation 22 in the form

$$T(x, z) = T_1 + T_0'(z - z_0) + T^*(x, z) \quad (24)$$

where $T^*(x, z)$ represents the effects of forced convection. Thus, in the exact result, $T^*(x, z)$ approaches zero as forced convection approaches zero. Furthermore, from the properties of the Mathieu functions, the function $T^*(x, z)$ has a certain symmetry about the midline. Specifically, when the midline occurs at $x = L/2$,

$$T^*(x = L, z) = -T^*(x = 0, z) . \quad (25)$$

Thus, for a symmetrical flow system, the increase in temperature above the conductive solution at a point $0 < x < L/2$ is just balanced by a decrease in temperature below the conductive solution at a mirror image point $L/2 < x < L$. Similar observations can be made for the geothermal gradient, that is

$$\frac{dT^*}{dz}(x = 0, z) = \frac{-dT^*}{dz}(x = L, z) . \quad (26)$$

It follows naturally that at the midline, $T^*(x = L/2, z) = 0$, indicating that the temperature distribution is a manifestation of pure conduction only. This results from the fact that $v_z(x = L/2)$ equals zero.

CONCLUSIONS

1. The factors that affect the steady-state temperature distribution in a homogeneous, isotropic regional flow system are: (a) the intrinsic properties of the medium and contained fluid, namely the thermal diffusivity of the solid-fluid complex and the hydraulic conductivity; (b) the water-table configuration; and (c) the ratio of basin depth to basin length. The influence of these factors is determined by solving an appropriate flow problem independently of temperature considerations, and incorporating the results in the conduction-convection equation.

2. The two-dimensional flow field is generally characterized by a midline separating a recharge area from a discharge area. The hydraulic gradient decreases with increasing depth in recharge areas, and increases with increasing depth in discharge areas. At the midline of flow, the hydraulic gradient is invariant with depth.

3. Convective heat losses on the recharge side of the midline of flow are just balanced by convective heat gains on the discharge side of the midline. Hence, any water-table configuration that tends to shift the midline either toward the valley bottom or topographic high will have a pronounced effect on the resulting temperature distribution. The nature of the flow as distributed on both sides of the midline dictates that the geothermal gradient increase with increasing depth in recharge areas and decrease with increasing depth in discharge areas. At the midline itself, the geothermal gradient is a manifestation of pure conduction.

4. Heat flow at the surface is a manifestation of the geothermal gradient, and is greater in ground-water discharge areas than in recharge areas.

5. A two-dimensional plot of the temperature distribution (such as given in Figs. 2 and 3) together with a plot of the potential distribution of the flow field provides all the pertinent information for a quantitative study of forced convection in regional ground-water flow. The conductive flux can be calculated from point to point along any streamline of heat conduction with Fourier's equation, $\mathbf{J} = -\kappa\nabla T$. The nonlinearity of the temperature gradient along any streamline of heat conduction except in the vicinity of the midline of fluid flow is obvious from Figure 4b. Hence, the conductive flux increases from the bottom of the basin to the water table along the flow line $x = 0$. Conversely, the conductive flux decreases from the bottom of the basin to the water table along the flow line $x = L$. In order that the total flux be conserved, a convective flux must be operative, and must accordingly increase along fluid flow lines in recharge areas, and decrease along fluid flow lines in discharge areas. The convective flux is easily calculated at any point in the system from $\mathbf{J}_c = \rho C\mathbf{v}T$, where $\mathbf{v}$ is velocity of fluid movement at the point (obtained from the regional flow pattern), and T is tempera-

ture at the point (obtained from the isotherm distribution).

6. A relative measure of the simultaneous transport of heat by the bulk motion of a fluid to that by pure conduction may be obtained from the dimensionless group $K\ell(z_0/L)/\alpha$, where ℓ is some critical length characteristic of fluid motion. For the cases examined, ℓ was found to be equivalent to the mean water-table elevation and to the maximum height of the water table, both with reference to z_0, and to the amplitude of water-table oscillation. The parameter ℓ is easily determined for any case by evaluating the coefficients of a smooth water-table configuration.

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REFERENCES CITED

Bird, R. B., Stewart, W. E., and Lightfoot, E. N., 1960, Transport phenomena: New York, John Wiley & Sons, Inc., 780 p.

Hubbert, M. K., 1940, The theory of ground-water motion: Jour. Geology, v. 48, p. 785–944.

Mifflin, M. D., 1968, Delineation of groundwater flow systems in Nevada: Reno, Nevada, Desert Research Inst. Tech. Rept. Ser. H-W, no. 4, 80 p.

Schneider, R., 1964, Relation of temperature distribution to ground-water movement in carbonate rocks of central Israel: Geol. Soc. America Bull., v. 75, p. 209–216.

Slattery, J. C., 1972, Momentum, energy, and mass transfer in continua: New York, McGraw-Hill Book Co., 679 p.

Stallman, R. W., 1963, Computation of ground-water velocity from temperature data, *in* Methods of collecting and interpreting ground-water data: U.S. Geol. Survey Water-Supply Paper 1544-H, p. 36–46.

Toth, J., 1962, A theory of groundwater movement in small drainage basins in central Alberta, Canada: Jour. Geophys. Research, v. 67, no. 11, p. 4375–4387.

—— 1963, A theoretical analysis of groundwater flow in small drainage basins: Jour. Geophys. Research, v. 68, no. 16, p. 4795–4812.

Van Orstrand, C. E., 1934, Temperature gradients, *in* Problems of petroleum geology: Tulsa, Okla., Am. Assoc. Petroleum Geologists, p. 989–1021.

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