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Growth Rate of a Penny-Shaped Crack in Hydraulic Fracturing of Rocks, 2

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The deformation and growth of a vertical penny-shaped crack, fractured hydraulically, is investigated when fluid is injected from an inlet at the center of the crack at a constant flow rate. The total flow rate at the inlet is divided into three parts: flow rate extracted from an outlet hole at an arbitrary distance above the center, fluid loss rate from the crack surface, and total fluid mass change in the crack. Two cases are considered: in case 1, inlet flow rate is initially greater than the sum of the outlet flow and fluid loss rates, and in case 2 the reverse holds true. Two subcases *a* and *b* are also considered, depending on the values of outlet pressure. Ranges of the inlet flow rate and the outlet pressure are discussed for which the crack attains stationary states and the fluid can be extracted continuously. Subcase *b*, where the outlet pressure is less than or equal to the difference between the tectonic stress and the fluid head at the inlet, is found to be more practical, and reasonable outlet flow rates are obtained in this case. It is also found that case 2*b* is preferable to case 1*b* to obtain the fluid with higher temperature. Results are expected to be of use in considerations of heat extraction from hot dry rock.

INTRODUCTION

The study of the heat extraction from a crack fractured by hydraulic pressure requires the solution of the appropriate equations of linear elastic fracture mechanics together with the appropriate equations of fluid mechanics and heat transmission. The equations of elasticity are required to establish the surface area and width of the crack.

Heat extraction problems on the fluid flowing in narrow spaces have been studied recently by *Bodvarsson* [1969], *Gringarten et al.* [1975], and *Lowell* [1976]. These analytical studies are confined to one-dimensional flow problems. The fluid flow in the crack should be treated at least as two-dimensional after drilling an outlet hole. Using a finite difference method, *Harlow and Pracht* [1972] and *McFarland* [1975] of the Los Alamos Scientific Laboratory analyzed two-dimensional models numerically. In most of the studies made thus far, however, the values of the crack radius and the crack width are assumed in advance, though they are functions of the fluid pressure and therefore of the flow rate.

The radius and width of the crack generally change with time for an arbitrarily given inlet flow rate. For example, in some cases the crack keeps expanding, and in other cases the crack tends to shut because of the large outlet flow rate. It is therefore of practical use to obtain the range for which the crack attains a stationary state and the water can be extracted continuously. In this paper the deformation and growth of a crack are investigated for that purpose.

The total flow rate at the inlet is divided into three parts: the flow rate which can be extracted from the outlet hole, the fluid loss rate, and the total mass change of the fluid in the crack. The fluid loss rate is assumed to be a linear function of the pressure in the crack. From the standpoint of available theoretical results it is reasonable to treat two-dimensional problems of a penny-shaped crack as a starting point for the analysis of a vertically oriented fracture.

In a previous paper the stable growth of a penny-shaped crack without outlet holes was investigated analytically [*Abé et al.*, 1976]. It was verified there for axisymmetrical problems that the fluid does not penetrate everywhere in the crack and that the classical solution by Sack is approximately valid for large cracks expanded under the injection of the fluid at a constant flow rate. The results there may be applicable to the present nonsymmetrical problem of a penny-shaped crack with outlet hole which is drilled after the crack attains its full growth. The examples to be presented subsequently are for the case when the flow rate at the inlet is constant, and ranges will be given for which the crack will be stationary.

FLUID FLOW IN A PENNY-SHAPED CRACK

We consider a penny-shaped crack having a radius R and width w (in the z direction; see Figure 1). Fluid is injected from the inlet at the center of the crack and removed in part at the outlet, $x = a$, where x is the distance measured in the vertical direction from the center. The radii of the inlet and outlet holes are denoted by R_0 and R_a, respectively.

The total mass flow rate at the inlet well bore can be divided into the following three parts:

$$q_0 = q_a + q_E + q_L \tag{1}$$

where q_a is the effective flow rate equal to the outlet flow rate, q_E is the total mass change in the crack, and q_L corresponds to the total fluid loss in the crack per unit time.

It was shown in the previous paper [*Abé et al.*, 1976] that the classical solution by *Sack* [1946] is approximately valid for cracks with large fracture radius when the crack is expanding under a constant flow rate. This means that the pressure distribution is almost independent of the fluid viscosity. By applying this result to the present problem the equations of linear momentum are found to be

$$\partial p/\partial r = -\rho_f g \cos\theta \qquad \partial p/\partial\theta = \rho_f g r \sin\theta \tag{2}$$

where p is the fluid pressure in the crack, g is the acceleration due to gravity, and ρ_f is the fluid density.

Equation (2) is easily integrated as

$$p(r, \theta) = p_0 - \rho_f g r \cos\theta \tag{3}$$

where p_0 is the fluid pressure at $r = 0$.

The fluid loss term is in general a function of the fluid pressure. Here it is assumed to be a linear function of p for simplicity; i.e.,

$$2\rho_f u_L = C_{L0} + C_{L1}(p_0 - \rho_f g r \cos\theta) \tag{4}$$

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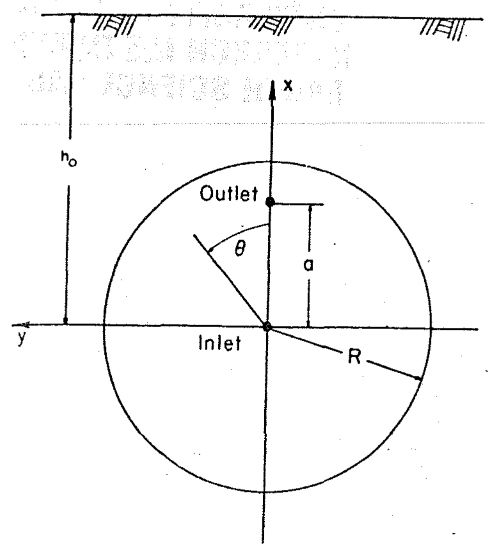

Fig. 1. Geometry and coordinate system.

:ere u_L is the fluid loss rate per unit area of fracture surface :C_{Lo} and C_{L1} are constants. It is easily seen from (4) that q_L :ned in (1) is expressed by

$$q_L = \pi R^2(C_{Lo} + C_{L1}p_0) \quad (5)$$

:he early stage of the growth of the crack this expression :ht have some error [*Hall and Dollarhide*, 1968].

:should be noted that a more refined form of (1) is

$$q_0 = q_a + q_E + q_L + q_T$$

:ere q_T is the flow rate due to the thermal contraction of the :k. In this paper, (1) is employed for simplicity, since q_T :y be small, although $\int_0^t q_T\, dt$ is large for large t.

STRESS INTENSITY FACTOR AND OPENING OF CRACK

Consider a vertical crack whose center is situated a distance :below the surface of the earth which does not interact with :crack. If the crack is absent, the compressive tectonic stress > 0) is acting across any vertical plane and can be divided :o a constant stress and a hydrostatic pressure. Hence if the :sity of the rock is ρ_s ($\rho_s > \rho_f$),

$$\sigma_z = -S = -(S_0 - K_a\rho_s g x) \quad (6)$$

:ere S_0 is the tectonic stress at $r = 0$ and K_a is the coefficient :active rock pressure. Thus the boundary condition of the :ack plane is

$$= -(p - S) = -(p_0 - S_0) - (K_a\rho_s - \rho_f)gr\cos\theta \quad (7)$$

$$\tau_{rz} = \tau_{\theta z} = 0$$

The case of a penny-shaped crack opened by non- :mmetrical pressure distribution was studied by *Keer* [1964]. :follows from his paper that for $r > R$ and $z = 0$,

$$\sigma_z = F^{(0)} + F^{(1)}\cos\theta \quad (8)$$

:here

$$F^{(n)} = \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{r^{n+1}} \frac{d}{dr} \int_0^R \frac{t^{2n+1}k_n(t)}{(r^2 - t^2)^{1/2}}\, dt \qquad n = 0, 1 \quad (9)$$

:nd

$$k_n(t) = \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{t^{2n}} \int_0^t \frac{s^{n+1}g_n(s)\, ds}{(t^2 - s^2)^{1/2}} \qquad n = 0, 1 \quad (10)$$

with

$$g_0(r) = -(p_0 - S_0) \qquad g_1(r) = -(K_a\rho_s - \rho_f)gr \quad (11)$$

The stress intensity factor near the tip of the crack is

$$K = \lim_{r\to R} (r - R)^{1/2}\sigma_z \quad (12)$$

Hence

$$K = -\frac{(2R)^{1/2}}{\pi} \int_0^1 \frac{\xi}{(1 - \xi^2)^{1/2}} \cdot [g_0(R\xi) + g_1(R\xi)\xi\cos\theta]\, d\xi \quad (13)$$

Introduction of (11) into (13) yields

$$K = \frac{(2R)^{1/2}}{\pi}\left[p_0 - S_0 + \frac{2}{3}gR\,(K_a\rho_s - \rho_f)\cos\theta\right] \quad (14)$$

The crack opening displacement $w(r, \theta)$ is given as follows [*Keer*, 1964]:

$$w(r, \theta) = \frac{4(1 - \nu^2)}{E}(G^{(0)} + G^{(1)}\cos\theta) \quad (15)$$

where E and ν are Young's modulus and Poisson's ratio, respectively, and

$$G^{(n)} = -\left(\frac{2}{\pi}\right)^{1/2} r^n \int_r^R \frac{k_n(t)\, dt}{(t^2 - r^2)^{1/2}} \qquad n = 0, 1 \quad (16)$$

By use of (10) and (11), (15) becomes

$$w(r, \theta) = \frac{8(1 - \nu^2)}{\pi E} \cdot \left[p_0 - S_0 + \frac{2}{3}(K_a\rho_s - \rho_f)gr\cos\theta\right](R^2 - r^2)^{1/2} \quad (17)$$

The total mass in the crack is given by

$$Q = \int_0^R \int_{-\pi}^{\pi} \rho_f w r\, d\theta\, dr \quad (18)$$

Equations (17) and (18) lead to

$$Q = (2\pi/D)[R^3(p_0 - S_0)] \quad (19)$$

where

$$D = 3\pi E/8(1 - \nu^2)\rho_f \quad (20)$$

The fracture toughness of the rock defined by K_C is assumed to be constant everywhere in the rock. In this paper it is assumed that the fracture criterion is expressed approximately by $\bar{K} \geq K_C$, where $\bar{K}$ is the average stress intensity factor introduced by the definition

$$\bar{K} = \frac{1}{2\pi}\int_{-\pi}^{\pi} K\, d\theta = \frac{(2R)^{1/2}}{\pi}(p_0 - S_0) \quad (21)$$

Then (19) is written as

$$Q = (2^{1/2}\pi^2/D)\bar{K}R^{5/2} \quad (22)$$

Furthermore, q_E, as defined in (1), is

$$q_E = dQ/dt \quad (23)$$

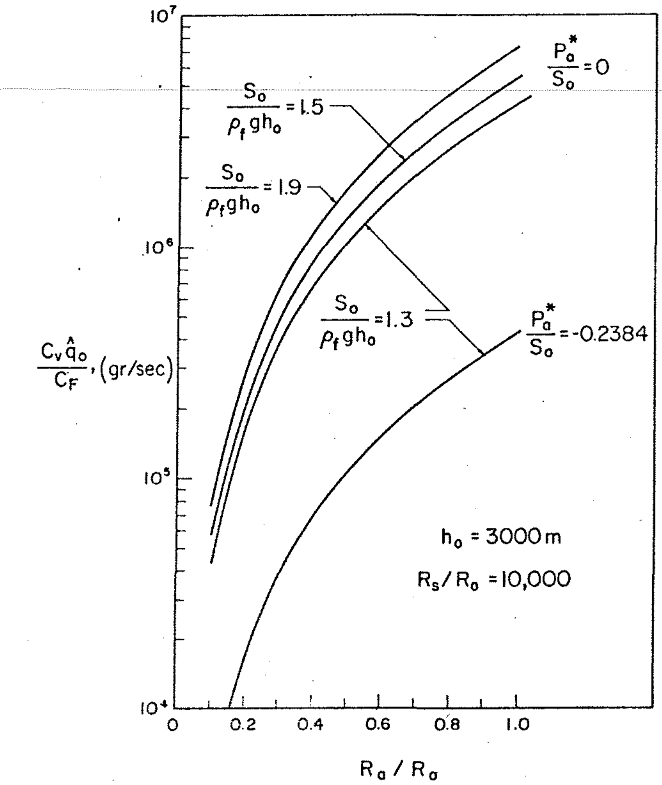

Fig. 2*a*. Relation between q_0 and R_a/R_0 for subcase *a* ($p_a^*/S_0 = 0$) and subcase *b* ($p_a^*/S_0 = -0.2384$).

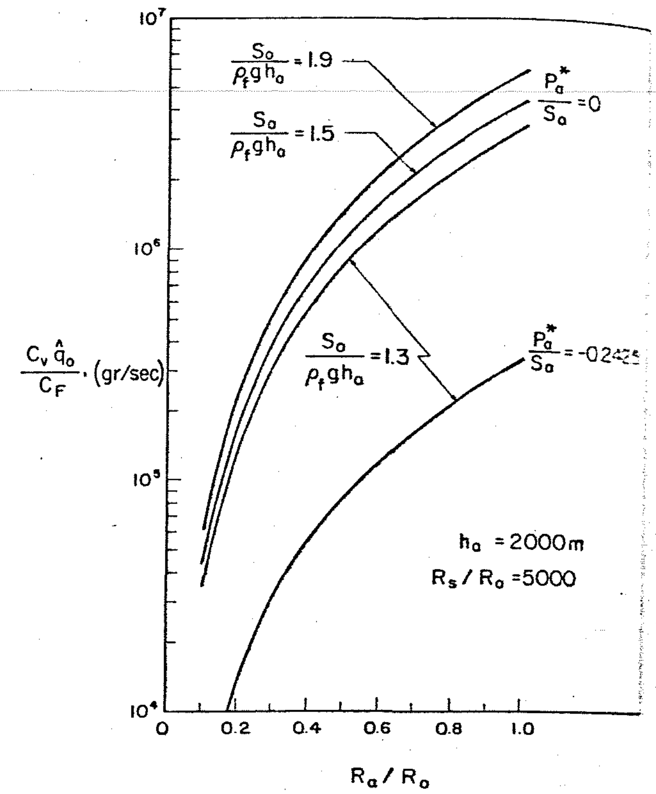

Fig. 2*b*. Relation between q_0 and R_a/R_0 for subcase *a* ($p_a^*/S_0 = 0$) and subcase *b* ($p_a^*/S_0 = -0.2425$).

When the outlet hole is provided, the crack radius is still expanding or ceases expanding, depending on the value of q_E. Therefore the following two cases are considered:

Case 1. If $q_0 - q_a - q_L = q_E \geq 0$ and therefore $dR/dt \geq 0$ and $d\bar{w}/dt \geq 0$, (23) and (22) yield

$$q_E = (5\pi^2/2^{1/2}D)\bar{K}R^{3/2}\, dR/dt \qquad (24)$$

Case 2. If $q_0 - q_a - q_L = q_E \leq 0$ and therefore $dR/dt = 0$ and $d\bar{w}/dt \leq 0$, (23) and (19) lead to

$$q_E = (2\pi/D)R^3\, dp_0/dt \qquad (25)$$

where $\bar{w}$ is the average value of crack width w. It can be seen that $dR/dt = 0$ for any case when $q_E = 0$ or $q_0 = q_a + q_L$. In case 2, $dp_0/dt = 0$ for $q_E = 0$ ($q_0 = q_a + q_L$), and $dp_0/dt < 0$ for $q_E < 0$ ($q_0 < q_a + q_L$).

Flow Rate and Fracture Radius

It is assumed that the following Bernoulli equation is applicable to the flow in the neighborhood of the throat of the outlet. Then

$$p_0 - g\rho_f a = -p_a^* + g\rho_f(h_0 - a) + \tfrac{1}{2}\rho_f(C_v v_a)^2 \qquad (26)$$

where v_a is the fluid velocity in the outlet well bore just outside the throat and the constant C_v (> 1) is an outlet head loss. The fluid density ρ_f has been assumed to be constant in both sides of the throat. A suction pressure by the outlet pump is denoted by p_a^*.

Since

$$q_a = \pi R_a^2 v_a \rho_f \qquad (2$$

where R_a is the outlet pump radius, it follows that

$$\frac{q_a}{2\pi} = \left(\frac{\rho_f}{2}\right)^{1/2} \frac{R_a^2}{C_v}(p_0 + p_a^* - \rho_f g h_0)^{1/2} \qquad (2$$

or

$$\frac{q_a}{2\pi} = \left(\frac{\rho_f}{2}\right)^{1/2} \frac{R_a^2}{C_v}\left(S_0 + \frac{\bar{K}\pi}{(2R)^{1/2}} - \rho_f g h_0 + p_a^*\right)^{1/2} \qquad (2)$$

by substituting p_0 from (21).

Equation (1) can be written in two ways, depending on the cases mentioned in the last section:

Case 1

$$q_0 - q_a - q_L = q_E \leq 0 \qquad dR/dt = 0 \qquad d\bar{w}/dt \leq 0 \qquad \text{at} \quad t_D = t_{DS}$$

$$q_0 = \underbrace{C_1\left(1 + \frac{K_D}{(R/R_0)^{1/2}} - \frac{1}{\Delta} + \frac{p_a^*}{S_0}\right)^{1/2}}_{q_a} + \underbrace{\frac{5}{2}K_D\left(\frac{R}{R_0}\right)^{3/2} q_0 \frac{d}{dt_D}\left(\frac{R}{R_0}\right)}_{q_E} + \underbrace{(B_0 + B_1)\left(\frac{R}{R_0}\right)^2 + K_D B_1\left(\frac{R}{R_0}\right)^{3/2}}_{q_L} \qquad (3$$

TABLE 1. Case 1, Where $q_0 > q_a(t_{DS}) + q_L(t_{DS})$

	q_0, $\times 10^4$ g/s	p_a^*/S_0	q_a, $\times 10^4$ g/s $t_D = t_{DS}$	q_a, $\times 10^4$ g/s $t_D = \infty$	$\Delta p_0/S_0$ $t_D = t_{DS}$	$\Delta p_0/S_0$ $t_D = \infty$	R/R_0 at $t_D = \infty$
A	2.013	−0.23852	8.247	5.096	0.24077	0.23938	13,500
B	1.718	−0.23882	7.671	5.195	0.24077	0.23971	12,500
C	1.451	−0.23916	6.965	5.305	0.24077	0.24009	11,500
D	1.954	−0.24172	8.008	4.948	0.24491	0.24294	6,750
E	1.668	−0.24215	7.449	5.045	0.24491	0.24342	6,250
F	1.409	−0.24264	6.763	5.151	0.24491	0.24396	5,750

For cases 1A–1C, R_s/R_0 = 10,000, h_0 = 3,000 m, and R_a/R_0 = 0.5.
For cases 1D–1F, R_s/R_0 = 5,000, h_0 = 2,000 m, and C_F = 2.0.

Case 2

$$q_0 - q_a - q_L = q_E \le 0 \qquad dR/dt = 0 \qquad d\bar{w}/dt \le 0 \quad \text{at} \quad t_D = t_{DS}$$

$$q_0 = \underbrace{C_1\left(\frac{p_0}{S_0} - \frac{1}{\Delta} + \frac{p_a^*}{S_0}\right)^{1/2}}_{q_a} + \underbrace{\left(\frac{R}{R_0}\right)^3 q_0 \frac{d}{dt_D}\left(\frac{p_0}{S_0}\right)}_{q_E} + \underbrace{\left(B_0 + B_1\frac{p_0}{S_0}\right)\left(\frac{R}{R_0}\right)^2}_{q_L} \tag{31}$$

where

$$K_D = \pi\bar{K}/(2R_0)^{1/2}S_0 \qquad t_D = (Dq_0/2\pi R_0^3 S_0)t \qquad \Delta = S_0/\rho_f g h_0 \tag{32}$$

with

$$B_0 = \pi R_0^2 C_{L0} \qquad B_1 = \pi R_0^2 S_0 C_{L1}$$

$$C_1 = \pi(2\rho_f^2 g h_0 \Delta)^{1/2}\left(\frac{R_a}{R_0}\right)^2 \frac{R_0^2}{C_v} \tag{33}$$

and $t_D = t_{DS}$ is the time at which the outlet is provided. Each case can be further divided into the following two subcases according to the magnitude of p_a^*:

Subcase *a*

$$p_a^*/S_0 > 1/\Delta - 1 \tag{34a}$$

Subcase *b*

$$p_a^*/S_0 \le 1/\Delta - 1 \tag{34b}$$

The capability of water extraction from the outlet depends on (34). Let us consider case 1. The term for q_a in (30) is always a real number and decreasing function of R. First, it is assumed that the loss rate q_L is small in relation to other quantities in (30). Since q_a is a decreasing function of R/R_0, the quantity dR/dt_D does not vanish when $q_0 > q_a(t_{DS})$, so that the crack keeps expanding with time if a small difference $q_0 - q_a$ exists at $t_D = t_{DS}$. On the other hand, if q_L is not relatively small and is expressed by (5), the term $q_L + q_a$ is an increasing function of R/R_0, so that dR/dt_D can be equal to zero for $t_D > t_{DS}$. In case 1*a* the water can be extracted continuously from the outlet. In case 1*b*, q_a becomes zero when R reaches a critical value R_c. For $R > R_c$ the water cannot be extracted from the outlet, although the crack may continue to expand with time. R_c is determined from

$$1 + \frac{K_D}{(R_c/R_0)^{1/2}} - \frac{1}{\Delta} + \frac{p_a^*}{S_0} = 0 \tag{35}$$

The differential equation (30) can be integrated for a given constant q_0. The solution is

$$t_D = t_{DS} + \frac{5\pi\bar{K}}{2(2)^{1/2}S_0R_0^3} \cdot \int_{R_S}^{R} R^{3/2}\,dR\,\Bigg\{1 - \pi(2\rho_f S_0)^{1/2}\frac{RR_a^2}{q_0 C_v} \cdot \left[1 + \frac{\pi\bar{K}}{(2R)^{1/2}S_0} + \frac{p_a^*}{S_0} - \frac{\rho_f g h_0}{S_0}\right]^{1/2} - \frac{\pi}{q_0}\left[(C_{L0} + C_{L1}S_0)R^2 + \frac{\pi}{2^{1/2}}\bar{K}C_{L1}R^{3/2}\right]\Bigg\}^{-1} \tag{36}$$

where the initial condition has been introduced as

$$R = R_s \quad \text{at} \quad t_D = t_{DS} \tag{37}$$

TABLE 2. Case 2, Where $q_0 < q_a(t_{DS}) + q_L(t_{DS})$

	q_a, $\times 10^5$ g/s	p_a^*/S_0	q_0, $\times 10^4$ g/s $t_D = t_{DS}$	q_0, $\times 10^4$ g/s $t_D = \infty$	$\Delta p_0/S_0$ $t_D = t_{DS}$	$\Delta p_0/S_0$ $t_D = \infty$
A	1.289	−0.23852	8.247	4.647	0.24077	0.23923
B	1.232	−0.23882	7.671	4.649	0.24077	0.23954
C	1.162	−0.23916	6.965	4.653	0.24077	0.23988
D	1.183	−0.24172	8.008	3.819	0.24491	0.24245
E	1.127	−0.24215	7.449	3.823	0.24491	0.24288
F	1.059	−0.24264	6.763	3.827	0.24491	0.24337

For cases 2A–2C, R_s/R_0 = 10,000, h_0 = 3,000 m, and R_a/R_0 = 0.5.
For cases 2D–2F, R_s/R_0 = 5,000, h_0 = 2,000 m, and C_F = 2.0.

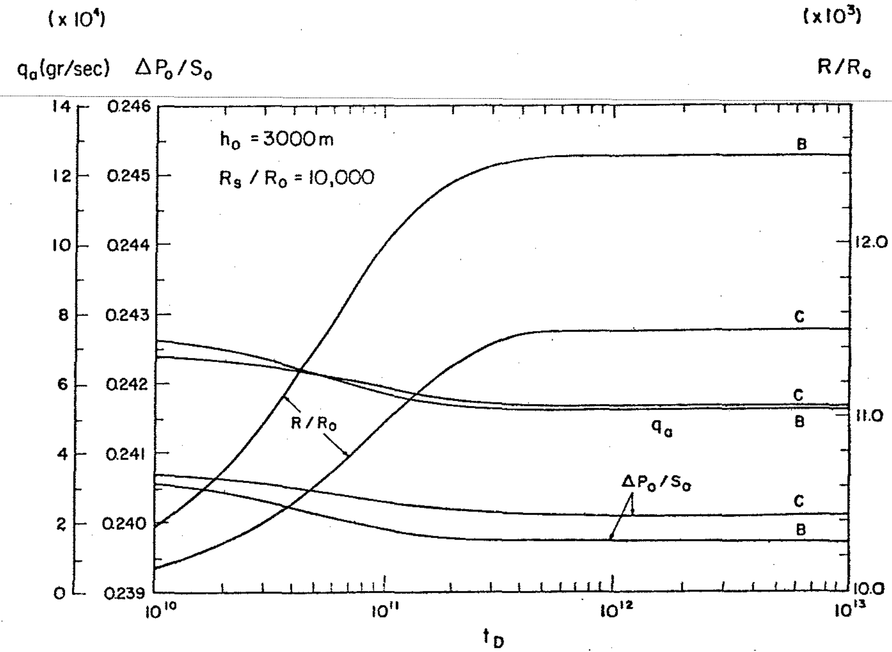

Fig. 3*a*. Flow rate q_0, pressure differential $p_0 - \rho_f g h_0$, and crack radius R/R_0 as a function of time t_D for case 1*b* (R_S/R_0 = 10,000).

The time range $t_D \leq t_{DS}$ corresponds to the initial fracturing state. The relation between R and t_D in this time range is obtained as

$$t_D = \frac{Dq_0}{2\pi S_0 R_0^3} t = \frac{5\pi \bar{K}}{2(2)^{1/2} S_0 R_0^3} \int_{R_0}^{R} R^{3/2}\, dR$$

$$\cdot \left\{ 1 - \frac{\pi}{q_0} \left[(C_{L0} + C_{L1} S_0) R^2 + \frac{\pi}{2^{1/2}} \bar{K} C_{L1} R^{3/2} \right] \right\}^{-1} \qquad (38)$$

For case 2, R is constant, but p_0 is changing with time. The relation between p_0 and t_D is easily obtained by integrating (31), where p_0 is a decreasing function of time. In order to have real values of q_a, p_0 must be in the range

$$p_0/S_0 > 1/\Delta - p_a^*/S_0 \qquad (39)$$

$$p_0/S_0 > 1 \qquad (40)$$

The last condition has been obtained from the condition $\tilde{w} >$ 0. Case 2*a* satisfies condition (39) automatically if condition (40) is satisfied. For case 2*b*, condition (39) gives a limitation for p_0. After p_0 reaches a critical value p_{0c}, the water cannot be extracted from the outlet. This critical value is determined by

$$p_{0c}/S_0 - 1/\Delta + p_a^*/S_0 = 0 \qquad (41)$$

When p_0 continues to decrease with time and reaches the value S_0, the width of the crack is reduced to zero, at least at the inlet, as shown in (17). In a strict sense, (17) must be modified, since negative values of the displacement are not permitted physically. The area with the negative displacement is large when $p_0 = S_0$, so that the solution of the problem with a

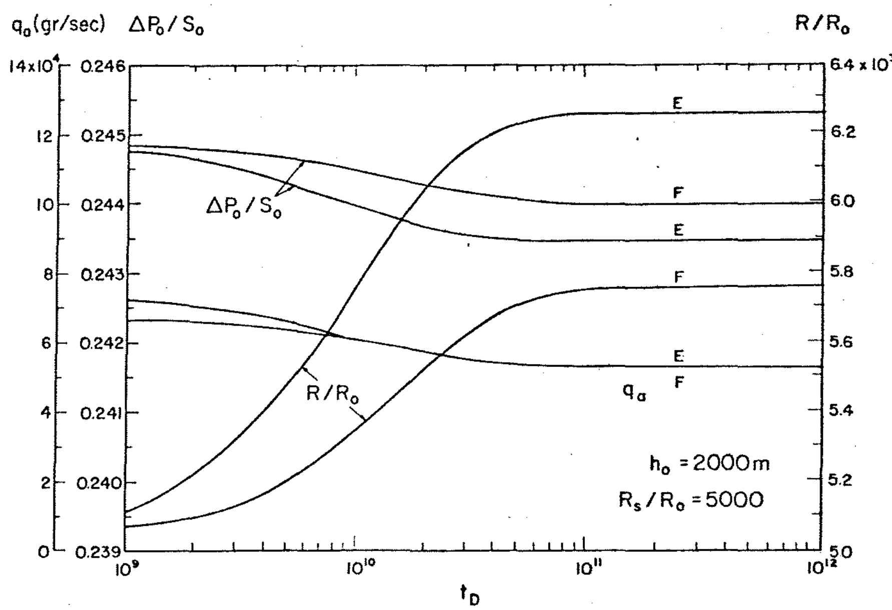

Fig. 3*b*. Flow rate q_0, pressure differential $p_0 - \rho_f g h_0$, and crack radius R/R_0 as a function of time t_D for case 1*b* (R_S/R_0 = 5000).

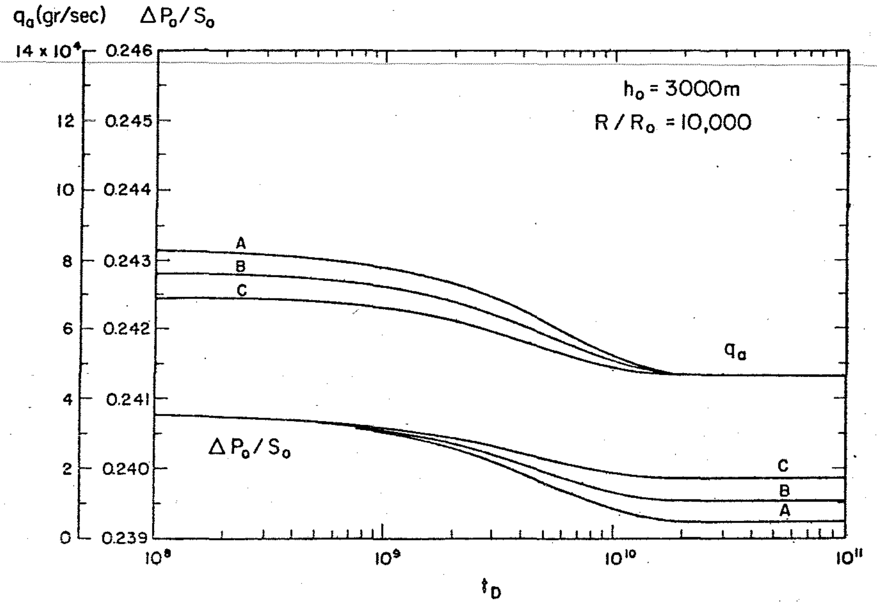

Fig. 4*a*. Flow rate q_0 and pressure differential $p_0 - \rho_f g h_0$ as a function of time t_D for case 2*b* (R/R_0 = 10,000).

moving boundary is required instead of (17). However, the effect of this area on (31) may be small unless p_0 is very close to S_0. When thermal contraction is not negligible, the crack will not close.

As has been stated above, it is necessary to introduce the fluid loss term in case 1 to design the hot water extraction from the stationary crack, i.e., the crack whose radius does not exceed a finite value. The loss term introduced here resulted from the definition of (4) or (5) and is an approximate expression. The rate q_L might be a function of time t even though R and p_0 are constant. When q_L is a slowly decreasing function of t [*Delisle*, 1975], the crack radius R increases and tends to infinity with small propagation velocity even if $dR/dt_D = 0$ at $t_D = t_{DS}$. In subcase *a* the water can be extracted continuously, while in subcase *b* it cannot for $R \geq R_c$, which is defined in (35). In case 2 the crack might close after a long period if the thermal contraction of the rock is negligible. However, if q_0 remains still constant, the pressure increases, and the crack will reopen. The exact expression of the loss rate for hot dry rocks has not yet been determined, and more systematic experiments may be required. The relation between R and t_D in case 1 depends largely on the loss rate term. On the other hand, in case 2 the crack radius remains constant, and the pressure p_0 may converge to a certain value regardless of the magnitude of q_L if (39) and (40) are satisfied, since both q_a and q_L are decreasing functions of p_0. If q_L decreased with time t after arriving at $dp_0/dt_D = 0$, the rate q_0 would be larger than $q_a + q_L$, and the pressure would increase. The relation between p_0 and t_D can be obtained from (31) also when $dp_0/dt_D > 0$ provided that $p_0(t_D) \leq p_0(t_{DS})$.

The flow rate q_a in subcase *b* is less than that in subcase *a*, and hence the fluid temperature in subcase *b* is expected to be high in comparison with that in subcase *a*. If a large quantity of hot water is required, the multiply fractured system should

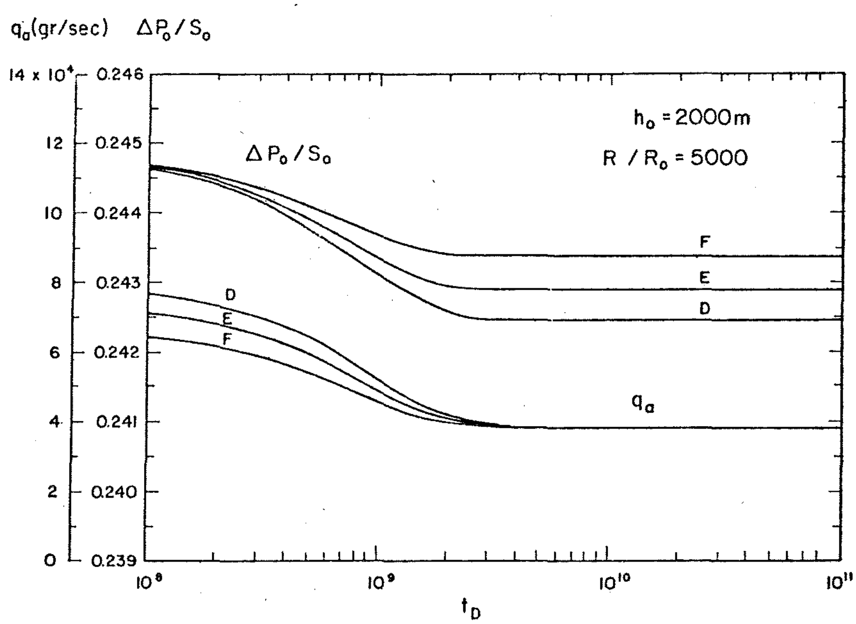

Fig. 4*b*. Flow rate q_0 and pressure differential $p_0 - \rho_f g h_0$ as a function of time t_D for case 2*b* (R/R_0 = 5000).

be adopted. The pressure p_a^* plays an important role in controlling the outlet flow rate in practical problems.

Numerical Examples

The dependency of q_0 on various parameters is studied when there is the condition of a stationary crack (dR/dt_D and $dp_0/dt_D = 0$). Equations (30) and (31) are rewritten as

$$q_0 = \hat{q}_0 = C_F q_a \quad \text{at} \quad t_D = t_{DS} \tag{42}$$

where C_F is a fluid loss term such that

$$q_L = (C_F - 1)q_a \tag{43}$$

The properties used in the calculations are the following: $\rho_f = 1$ g/cm³, $[K_D]_{\bar{K}=K_C} = 1.118$, $R_0 = 10$ cm, and $C_v = 1.25$. The ratio $S_0/\rho_f g h_0$ has been assumed as 1.3, 1.5, and 1.9. For example, if $S_0 = K_a \rho_s g h_0$, where K_a is the coefficient of active rock pressure, then $S_0/\rho_f g h_0 = 1.3$ for $\rho_s = 2.65$ g/cm³ and $K_a = 0.49$. The results are shown in Figures 2*a* and 2*b*. The curves for $p_a^*/S_0 = 0$ correspond to subcase *a*, and those for $p_a^*/S_0 = -0.2384$ and -0.2425 correspond to subcase *b*. It is seen from these figures that q_0 depends largely on R_a/R_0. Values of mass flow rate employed so far in the calculations and experiments are fairly small. For example, $q_0 = 1.44 \times 10^5$ g/cm³ is used by *McFarland* [1975]. Relatively small values of R_a/R_0 and/or the compressive pressure p_a^* are required even when there is no fluid loss, i.e., when $C_F = 1.0$.

Actual mass flow rates at the inlet are usually above or below the $\hat{q}_0$ curve at $t_D = t_{DS}$. It follows from Figure 2 and (42) that q_E for subcase *a* is larger than that of subcase *b*. In order to extract the fluid with higher temperature, subcase *a* is not necessarily advantageous, as was stated in the previous section. Then two examples are calculated: cases 1*b* and 2*b*.

Equations (30) and (31) are integrated for given values of q_0 and p_a^*/S_0, where $R_a/R_0 = 0.5$, C_F is assumed as 2.0, and B_1 is taken as zero, since the effect of the fluid pressure on the fluid loss is small (as has been discussed by *Hall and Dollarhide* [1964]). Numerical data and the results are given in Tables 1 and 2, in which the following quantity is employed:

$$\Delta p_0/S_0 = p_0/S_0 - 1/\Delta \tag{44}$$

Case 1B, 1C, 1E, and 1F in Table 1 are graphed in Figures 3*a* and 3*b*, and all cases in Table 2 are graphed in Figures 4*a* and 4*b*. It is seen from these figures that q_a, $\Delta p_0/S_0$, and R/R_0 converge to stationary values within $t_D < 10^{10} \sim 10^{12}$, depending upon the numerical data. As has been stated previously, smaller values for the flow rates q_a and q_0 are required in order to get the fluid with higher temperature. In all cases treated in Tables 1 and 2 and also in Figures 3 and 4 those flow rates are fairly small. The flow rate q_a is proportional to $(p_a^* + \Delta p_0)/S_0$, and this sum is found to be very small in comparison with each quantity. It is therefore required for the effective working of the actual geothermal system that the pressure p_a^* be carefully controlled.

Furthermore, it is desirable that the distance a ($<R$) be close to R, since the fluid temperature is expected to be higher near the edge of the crack. Case 2*b* is preferable to case 1*b* in this regard, since a/R decreases with respect to time in case 1.

Conclusions

The deformation and growth of a vertical penny-shaped crack has been investigated when the fluid is injected from an inlet at the center of the crack and extracted in part from an outlet at an arbitrary distance above the inlet. Four cases have been considered: cases 1 and 2, with respect to the total mass change in the crack, and subcases *a* and *b*, which depend on the suction pressure of the outlet well bore. The conclusions are summarized as follows:

1. In case 1, $dR/dt \geq 0$ (Equation (24)), and in case 2, $dR/dt \equiv 0$ (equation (25)).
2. The possibility of obtaining a stationary crack ($dR/dt \to 0$) at $t \to \infty$ in case 1 depends largely on the fluid loss term.
3. If $dR/dt = 0$ ($R < h_0$) at $t \to \infty$, the water can be extracted continuously from the outlet in case 1*a*. In case 1*b* the water can also be extracted for $R < R_c$, where the critical radius R_c is given in (35).
4. In case 2*a* the water can be extracted continuously, whereas in case 2*b* the same holds true if $p_0 \geq p_{0c}$, where the critical pressure p_{0c} is determined by (41).
5. Subcase *b* is expected to be more practical than subcase *a* to obtain the water with higher temperature (Figures 2*a* and 2*b*).
6. The pressure p_a^* should be carefully controlled for the effective working of the actual geothermal system. Furthermore, case 2*b* is preferable to case 1*b* to obtain the water with higher temperature (Tables 1 and 2, Figures 3 and 4).

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