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Vertical Gradients of Heat Production in the Continental Crust

1. Theoretical Detectability from Near-Surface Measurements

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The linear relation between heat flow and heat production suggests that in a gross sense the vertical distribution of crustal heat production beneath granitic rocks has a simple generalized form. Knowledge of the vertical gradient of heat production near the surface would permit selection between alternative simple models. However, attempts to determine a generalized gradient from heat-production measurements from boreholes are complicated by the occurrence of inhomogeneities on all observable scales. Variations in heat production typically observed on the hand-sample scale preclude meaningful estimates (even of the sign) of the generalized gradient in holes a few hundred meters deep, the depth typically drilled for heat-flow–heat-production measurements. In holes 3 km deep, the uncertainty in the gradient due to small-scale perturbations is generally reduced to acceptable levels with 100 or so samples. However, perturbations with wavelengths greater than 1 km and amplitudes sufficiently small to permit the linear heat-flow relation, can still preclude meaningful estimates of gradient if the phase is unfavorable. Confident determinations of the trend of heat production with depth in granitic rock will require observations in several holes to depths of a few kilometers or in very large numbers of holes drilled to lesser depths.

It is generally believed that a substantial fraction of the heat escaping from the earth's continental surfaces is generated by the radioactive decay of uranium, thorium, and potassium in the earth's crust. As heat flow from the earth's surface is an integrated effect of underlying sources, by itself it provides no direct information on how this heat production might be distributed vertically. It does, however, provide a basic constraint for geochemical studies of various kinds that clearly indicate a general upward concentration of sources in the crust [see e.g., *Heier and Adams*, 1965; *Lambert and Heier*, 1967, 1968*a*, 1968*b*; *Hyndman et al.*, 1968].

The recently discovered linear relation between heat flow and heat production in plutons, first described by *Birch et al.* [1968] and elaborated by *Roy et al.* [1968] and *Lachenbruch* [1968, 1970], provides new information on the vertical distribution of crustal sources, in certain regions that have undergone plutonic activity at least. In this relation the measured heat flow q is related to the measured heat production in near-surface plutonic rock $A(0)$ by

$$q = q^* + DA(0) \quad (1)$$

where q^* and D evidently are relatively constant over large geographic provinces [*Roy et al.*, 1968]. The parameter q^*, which has the dimensions of heat flow, is most simply identified with a uniform flux at depth. The parameter D has the dimension of depth, and it evidently relates to the vertical distribution of heat production in the region above the depth (z^*), at which the flux is uniform.

Relation 1 does not determine the heat-source distribution uniquely; three of the endless number of distributions permitted [see *Lachenbruch*, 1970, equation 4] are shown in Figure 1. Although the step function (Figure 1*a*) has been favored as an interpretive model largely because of its simplicity [*Birch et al.*, 1968; *Roy et al.*, 1968], it has been shown that the exponential model (Figure 1*c*) is the only one that would permit the validity of the empirical relation (1) in regions of differential erosion [*Lachenbruch*, 1968, 1970]. The proper selection between permissible models may be important to an understanding of the evolution of the crust and to estimates of mantle heat flow. It will also affect estimates of crustal temperature to some extent.

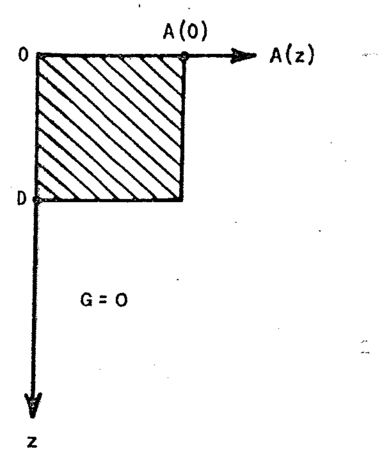
If the true source distribution were a simple, smooth one-dimensional one like those depicted

in Figure 1, a measurement of the dient of heat production near the help determine its form by indepe For convenience we define a norma of heat production

$$G(A) \equiv (-1/A)(dA/$$

where it is understood that G is e the earth's surface. For the step ure 1*a*) G = 0, and for the decreas tribution (Figure 1*b*) $G = (2D)$ creasing functions that are concav depth axis have $G < (2D)^{-1}$ ar are convex have $G > (2D)^{-1}$. In the exponential distribution (Figu D^{-1}. (It should be noted that the st the exponential model do not in an sent limiting cases of source distr mitted by equation 1.) Thus, idea nation of G from measurements boreholes in plutonic rocks taken w of D determined from relation information on the form of the v distribution in the hypothetical cr depth z^*. The relation between the models of Figure 1 is shown i

From observations available to c dently 9 or 10 km in the Sierra Basin and Range provinces [*Roy Lachenbruch*, 1968] and perhaps



a

Step function

Fig. 1. Three simple heat-pro
G is th

in Figure 1, a measurement of the vertical gradient of heat production near the surface would help determine its form by independent means. For convenience we define a normalized gradient of heat production

$$G(A) \equiv (-1/A)(dA/dz) \qquad (2)$$

where it is understood that G is evaluated near the earth's surface. For the step function (Figure 1*a*) G = 0, and for the decreasing linear distribution (Figure 1*b*) $G = (2D)^{-1}$. Simple decreasing functions that are concave toward the depth axis have $G < (2D)^{-1}$ and those that are convex have $G > (2D)^{-1}$. In particular, for the exponential distribution (Figure 1*c*), $G = D^{-1}$. (It should be noted that the step model and the exponential model do not in any sense represent limiting cases of source distributions permitted by equation 1.) Thus, ideally, a determination of G from measurements of A in deep boreholes in plutonic rocks taken with the value of D determined from relation 1 could yield information on the form of the vertical source distribution in the hypothetical crustal layer of depth z^*. The relation between G and D for the models of Figure 1 is shown in Table 1.

From observations available to date, D is evidently 9 or 10 km in the Sierra Nevada and Basin and Range provinces [*Roy et al.*, 1968; *Lachenbruch*, 1968] and perhaps 6 or 8 km in the eastern United States and more stable regions [*Birch et al.*, 1968; *Roy et al.*, 1968]. (*Jaeger* [1970] has recently reported a preliminary value of 4.5 km from three points on the Australian shield.) It has been shown [*Lachenbruch*, 1970] that z^* is likely to be of the order of D or larger. Hence the depth to which we are attempting to determine $A(z)$ is very large relative to the depth of boreholes and mines, or to the height of topographic relief, which might provide opportunities for direct sampling in plutonic rocks. Furthermore, variations in heat production are known to occur in plutonic rocks on every observable scale [see e.g., *Tilling et al.*, 1970]. In view of these facts, the question arises whether it is possible to obtain significant information on the general form of $A(z)$ from the measurement of G in boreholes; we are also led to ask whether the failure to observe a particular trend can be used as a valid argument against a particular model, or the observation of a predicted trend can be used as an argument for it. This paper addresses these questions.

Some Constraints Imposed by the Linear Relation

It is possible that the step distribution (Figure 1*a*) could be responsible for the linear relation (1) at one locality, the linear distribution

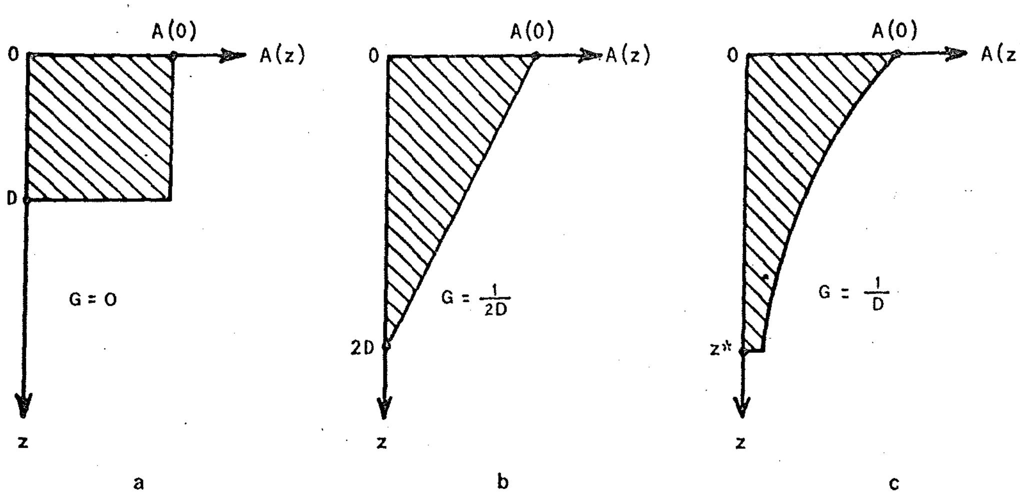


Fig. 1. Three simple heat-production models consistent with the linear heat-flow relation. G is the normalized gradient at the surface.

TABLE 1. Relation between Normalized Near-Surface Gradient $G(\phi)$ and Simple Models $\phi(z)$ for the Distribution of Heat Production with Depth

	Step Model		Linear Model		Exponential Model	
D, km	G, km^{-1}	G^{-1}, km	G, km^{-1}	G^{-1}, km	G, km^{-1}	G^{-1}, km
10	0	∞	0.050	20	0.10	10
7½	0	∞	0.067	15	0.13	7½
5	0	∞	0.10	10	0.20	5

See Figure 1.

(Figure 1*b*) at a second, and the exponential (Figure 1*c*) at a third. However, it seems much more likely that there is one idealized form for the heat production with depth, constrained by geochemical considerations, that is universally responsible for the validity of (1). Perhaps it is one of those shown in Figure 1 or perhaps it is some other. We shall assume such an idealized distribution exists and denote it by $\phi(z)$. Let $A(z)$ represent the actual measurable value of heat production at any depth z beneath a point on the surface. In general the difference between these two quantities will be responsible (in part) for departures of the measured heat flow q and heat production $A(0)$, from the idealized relation (1). This difference will be denoted by $\epsilon(z)$. Hence

$$A(z) = \phi(z) + \epsilon(z) \qquad (3)$$

In establishing the linear relation, the heat flow was measured in holes to some depth l (generally a few hundred meters), one or two orders of magnitude less than the depth z^* to which the relation $\phi(z)$ is expected to apply, i.e.,

$$l \ll D \lesssim z^* \qquad (4)$$

The surface heat production $A(0)$, equation 1, is generally identified with the mean heat production obtained from samples in such holes. Departures of $A(z)$ from $\phi(z)$ would generally result in departures $(\Delta q, \Delta A)$ of the $(q, A(0))$ point from the idealized values as follows:

$$\Delta q = \int_0^{z^*} \epsilon(z')\, dz' \qquad (5)$$

where z' represents depth beneath the ground surface, and

$$\Delta A = \frac{1}{l}\int_{-l/2}^{l/2} \epsilon(z)\, dz \qquad (6)$$

where origin of z is taken as the midpoint of the hole of depth l. (The depth variable z will be used in this sense hereafter.)

Barring large scale systematic departures, (1) is much more sensitive to the effect of ϵ on ΔA than on Δq. For example, if $\epsilon(z)$ were a sine wave with amplitude b and wavelength λ, the maximum value of ΔA would be b for wavelengths greater than l, whereas the maximum value of Δq would be $\lambda b/\pi$. Hence

$$\frac{\Delta A}{\phi(0)} \lesssim \frac{b}{\phi(0)} \qquad (7)$$

$$\frac{\Delta q}{q - q^*} \lesssim \frac{\lambda}{\pi D}\, \frac{b}{\phi(0)} \qquad (8)$$

Thus only for wavelengths λ approaching D (on the order of 5 or 10 km) is the measured value of q likely to be sensitive to perturbations unless their amplitude b is very large (equation 8). However, according to (6) and (7) the linear heat-flow relation could be obscured by perturbations $\epsilon(z)$ of any scale (exceeding the sampled interval) unless

$$b \ll \phi(0) \qquad (9)$$

Thus the observability of the linear relation (1) implies that the departure $\epsilon(z)$ of heat production from the idealized distribution $\phi(z)$ be substantially less than the mean surface heat production on scales exceeding the interval over which heat production is sampled. If we denote this mean heat production by A_m (instead of $A(0)$) where

$$A_m = \frac{1}{l}\int_{-l/2}^{l/2} A(z)\, dz \qquad (10)$$

the observability of the linear relation seems to imply that generally

$$\Delta A \ll \phi(0) \simeq A_m \qquad (11)$$

We shall next consider whether
small enough to allow the observ
linear relation may still be so large
the determination of the ideali
$G(\phi)$ from measurements of $A(z)$

DEPARTURES OF LARGER WAVE

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$$A' = \frac{\int_{-l/2}^{l/2} zA\, dz}{\int_{-l/2}^{l/2} z^2\, dz} = \frac{12}{l^3}\int_{-l/2}^{l/2} z$$

From (3), (10), and (13) it is se
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tion to departures from ϕ.

$$\phi(z) \simeq \phi_m + \phi' z$$

$$\epsilon(z) \simeq \Delta A + \Delta A' z$$

where ϕ_m and ϕ' are defined by
analogous to (10) and (13), ΔA
(6), and $\Delta A'$ is given by

$$\Delta A' = \frac{12}{l^3}\int_{-l/2}^{l/2} z\epsilon\, dz$$

Hence

$$A_m = \phi_m + \Delta A$$

$$A' = \phi' + \Delta A'$$

If $\epsilon(z)$ is represented by a
series, any term ϵ_ν of wave number

$$\epsilon_\nu = a_\nu \sin\frac{2\pi\nu z}{l} + b_\nu \cos$$

We shall next consider whether perturbations small enough to allow the observability of the linear relation may still be so large as to preclude the determination of the idealized quantity $G(\phi)$ from measurements of $A(z)$ in boreholes.

DEPARTURES OF LARGER WAVE LENGTH

A logical way to estimate the vertical gradient of heat production from several measurements of A, throughout a borehole of depth l, would be to perform a linear regression analysis to obtain a result of the form

$$A(z) \simeq A_m + A'z \qquad -\frac{l}{2} < z < \frac{l}{2} \tag{12}$$

where A_m is the mean value of heat production, and A' is defined as the mean gradient over the interval sampled. In considering departures of larger wavelength we shall assume that the sampling is sufficiently dense to characterize A as a continuous function of depth. In this case A_m would be as defined in equation 10, and A' would be given by

$$A' = \frac{\int_{-l/2}^{l/2} zA\,dz}{\int_{-l/2}^{l/2} z^2\,dz} = \frac{12}{l^3}\int_{-l/2}^{l/2} zA\,dz \tag{13}$$

From (3), (10), and (13) it is seen that (12) can be considered as the sum of the linear approximation to ϕ and the linear approximation to departures from ϕ.

$$\phi(z) \simeq \phi_m + \phi'z \tag{14}$$

$$\epsilon(z) \simeq \Delta A + \Delta A'z \tag{15}$$

where ϕ_m and ϕ' are defined by expressions analogous to (10) and (13), ΔA is defined by (6), and $\Delta A'$ is given by

$$\Delta A' = \frac{12}{l^3}\int_{-l/2}^{l/2} z\epsilon\,dz \tag{16}$$

Hence

$$A_m = \phi_m + \Delta A \tag{17}$$

$$A' = \phi' + \Delta A' \tag{18}$$

If $\epsilon(z)$ is represented by a trigonometric series, any term ϵ_ν of wave number ν is given by

$$\epsilon_\nu = a_\nu \sin\frac{2\pi\nu z}{l} + b_\nu \cos\frac{2\pi\nu z}{l} \tag{19}$$

where a_ν and b_ν are the amplitudes of the odd and even components respectively, and ν (the number of waves in a hole of length l) can assume fractional values. The wavelength λ of the perturbation is defined by $\lambda = l/\nu$. Substitution of (19) into (6) and (16) yields expressions for the departure of mean heat production ΔA and the departure in the mean gradient $\Delta A'$, caused by a departure of wave number ν from the idealized distribution.

$$\Delta A = \frac{b_\nu \sin \pi\nu}{\pi\nu} \tag{20}$$

$$\Delta A' = \frac{a_\nu}{l} W(\nu) \tag{21a}$$

where

$$W(\nu) = \frac{6}{\pi\nu}\left[\frac{\sin \pi\nu}{\pi\nu} - \cos \pi\nu\right] \tag{21b}$$

These relations are illustrated in Figures 2 and 3.

As anticipated in the previous section, if the even component has a wavelength greater than the hole depth l, the value of ΔA rapidly approaches the amplitude b_ν. For shorter wavelengths the amplitude can, of course, be substantial without significantly affecting the mean value (Figure 2).

We are mainly interested in the normalized gradient $G(\phi)$ and its approximation $G(A)$, which are represented as follows:

$$G(\phi) = \phi'/\phi_m \tag{22}$$

$$G(A) = A'/A_m \tag{23a}$$

$$= \left[G(\phi) \middle/ 1 + \frac{\Delta A}{\phi_m}\right] + \Delta A'/A_m \tag{23b}$$

By (11), the departure of the first term on the right in (23b) from $G(\phi)$ is of second order. We therefore take the second term of (23b) as a measure of the departure of $G(A)$ from the idealized value $G(\phi)$ and denote this departure by δG. Thus for the sinusoidal departure (19)

$$\delta G = \frac{\Delta A'}{A_m} = \frac{a_\nu}{A_m l} W(\nu) \tag{24}$$

The observability of (1) suggests that a_ν/A_m will be substantially less than unity (inequality

11). Even so, it is seen from Figure 3 that for perturbations on the order of the hole depth ($\nu \sim 1$) in holes typically drilled for heat-flow studies ($l \sim 0.3$ km), we can expect (24) $\delta G \sim 1$ km^{-1}. In such cases the perturbation δG is greater by an order of magnitude than the quantity sought, $G \sim D^{-1} \sim 0.1$ km^{-1} (see Table 1).

Thus small-amplitude perturbations of larger wavelengths can clearly preclude meaningful determinations of G in shallow boreholes. However, because their wavelength is not small relative to the interval of observation l, there is no satisfactory way of identifying them. Indeed, our best source of information regarding such perturbations might ultimately be the validity of (1) and the restriction it imposes (11). At the other end of the spectrum are very small-scale irregularities in A that introduce more predictable uncertainties in the determination of G. It is useful to consider them separately.

RANDOM SMALL-SCALE PERTURBATIONS

Having recognized the overwhelming effects that small-amplitude perturbations of moderate wavelength can have on the estimation of $G(\phi)$ in shallow boreholes, we shall ignore them for the moment and see what can be inferred from the more observable small-scale perturbations.

When the uranium, thorium, and potassium contents are measured in two adjacent hand specimens taken from an outcrop or core of typical granitic rock, the computed heat productions commonly differ by 20% or more (see σ^*/A_m^*, Table 1, *Lachenbruch and Bunker* [1971]). In this section we shall assume that such small-scale variations constitute the only departure of A from ϕ. Their effects on the estimate $G(A)$ of the idealized quantity $G(\phi)$ will be investigated.

In this case we have n samples taken at depths z_i, $i = 1, 2, 3, \ldots n$, extending over the length of the hole. The origin of z is again taken as the midpoint of the interval of observation, normally at a depth $l/2$ beneath the ground surface. The perturbations are given by

$$\epsilon(z_i) = A(z_i) - \phi(z_i) \tag{25}$$

We assume that $\epsilon(z_i)$ is randomly distributed so that ϕ is approximated by the regression line through (A_i, z_i)

$$\phi(z) \simeq [A_m \pm \alpha] + [A' \pm \beta]z \tag{26}$$

where α and β are the standard errors of A_m and A', respectively.

Denoting by σ the root-mean-square deviation of $A(z_i)$ from the regression line (26), we have

$$\sigma = \frac{1}{n-2}\left\{\sum_i^n [A_i - \phi(z_i)]^2\right\}^{1/2} \tag{27}$$

where $\phi(z_i)$ is the value given by (26) at the depth z_i,

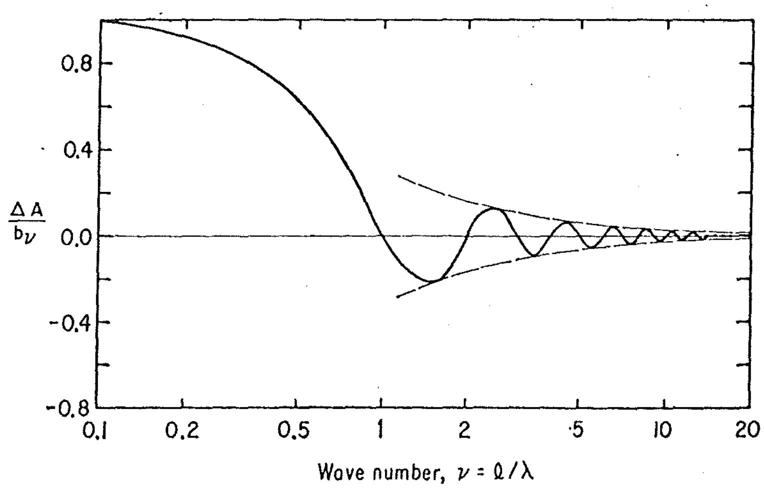

Fig. 2. Relation between the wave number ν of a cosine perturbation of amplitude b_ν and the error ΔA it causes in the mean heat production in a hole of depth l.

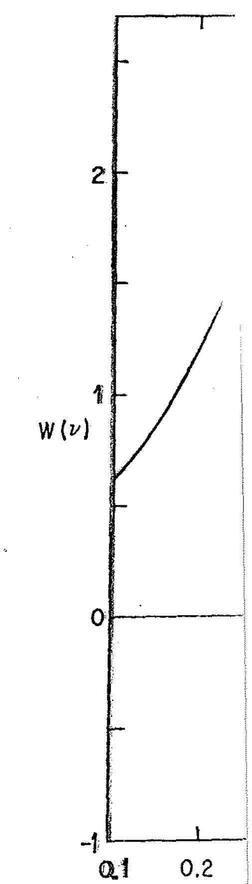

Fig. 3. Relation between th
error $\Delta A'$ it causes in

and $\alpha = \sigma/(n)^{1/2}$

$\beta = \sigma / \{\sum_i^n z_i^2\}$

If the n samples are equally
interval of length l, it can be
formula for the sum of consecu
tegers that

$\left\{\sum_i^n z_i^2\right\}^{1/2}$

$= l[n/12]^{1/2}[1 + 3/2n + O($

For samples large enough to be
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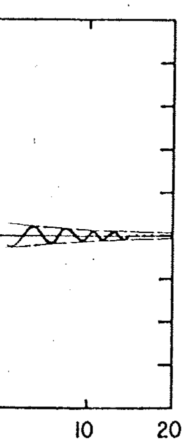

ation of amplitude b_ν and
hole of depth l.

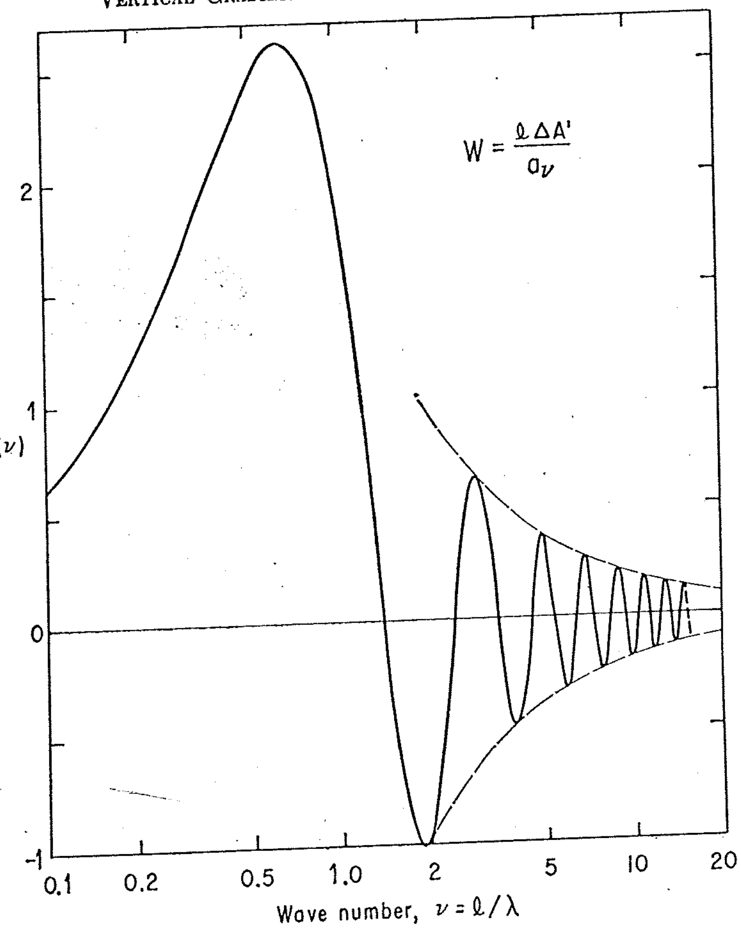

Fig. 3. Relation between the wave number ν of a sine perturbation of amplitude a_ν and the error $\Delta A'$ it causes in the mean gradient of heat production in a hole of depth l.

$$\alpha = \sigma/(n)^{1/2} \qquad (28)$$

and

$$\beta = \sigma \Big/ \Big\{\sum_i^n z_i^2\Big\}^{1/2}$$

If the n samples are equally spaced over the interval of length l, it can be shown by the formula for the sum of consecutive squared integers that

$$\Big\{\sum_i^n z_i^2\Big\}^{1/2} = l[n/12]^{1/2}[1 + 3/2n + O(1/n^2)] \qquad (29)$$

For samples large enough to be significant the higher order terms in (29) can be neglected. Hence we shall use

$$\beta = (12/n)^{1/2}\sigma/l \qquad (30a)$$

$$\beta = (12)^{1/2}\alpha/l \qquad (30b)$$

Thus insofar as random small-scale fluctuations are concerned, the uncertainty in determining the slope of the heat-production curve in a 3-km hole with ten samples is the same as that in a 0.3-km hole with 1,000 samples. In reality, the deeper, sparsely sampled hole should give a more satisfactory estimate because of the diminishing effects of larger wavelength perturbations with increasing hole depth, i.e., with increasing wave number ν (see Figure 3).

We are interested in determining the normalized gradient $G(A)$. Denoting its standard error by γ we have

$$G(A) \pm \gamma = (A' \pm \beta)/(A_m \pm \alpha) \qquad (31)$$

and

$$\gamma^2/G(A)^2 = \alpha^2/A_m^2 + \beta^2/A'^2 \qquad (32)$$

Combining (28), (30), and (32) yields

$$\gamma = (\sigma/lA_m)(12/n)^{1/2}(1 + l^2G^2/12)^{1/2} \quad (33)$$

As $G \lesssim D^{-1}$ (Table 1), by (4) the term in l^2G^2 can be neglected. Hence the standard error in the estimate of G is taken to be

$$\gamma = (\sigma/lA_m)(12/n)^{1/2} \quad (34a)$$

$$\gamma = \beta/A_m \quad (34b)$$

The validity of the foregoing analysis depends on the assumption that $\epsilon(z_i)$ has a normal frequency distribution. It is well known, however, that random variations in the distribution of trace elements are usually described better by a log-normal distribution (see e.g., *Rogers and Adams* [1963]). To accommodate this refinement we can rewrite equation 26

$$\phi(z) \simeq [A_m \pm \alpha][1 - (G(A) \pm \gamma)z] \quad (35)$$

By inequality (4) we may add terms of higher degree in Gz to the approximation without significantly affecting the linear terms. Hence (35) could be replaced with the approximation

$$\phi(z) \simeq [A_m^* \pm \alpha^*] \cdot \left\{1-(G^*\pm\gamma^*)z+\frac{1}{2!}(G^*\pm\gamma^*)^2z^2-\cdots\right\} \quad (36a)$$

$$\phi(z) \simeq [A_m^* \pm \alpha^*] \exp[-(G^* \pm \gamma^*)z] \quad (36b)$$

Thus fitting the exponential function (36b) to (A_i, z_i) is not significantly different from fitting a straight line to the same data in these applications. However, the starred quantities in (36b) can be identified with the parameters of a linear regression analysis of $(\ln A_i, z_i)$, which is based on the assumption of log normally distributed A_i.

$$\ln\phi \cong \ln(A_m^* \pm \alpha^*) - [G^* \pm \gamma^*]z \quad (37a)$$

$$\cong \ln A_m^* \pm \frac{\alpha^*}{A_m^*} - [G^* \pm \gamma^*]z$$

$$\alpha^* \ll A_m^* \qquad G^*l \ll 1 \quad (37b)$$

The standard error α^* of A_m^* and the standard error γ^* of G^* for a regression line through log normally distributed A_i are given approximately by expressions corresponding to (28) and (34)

$$\alpha^* = \sigma^*/(n)^{1/2} \quad (38)$$

$$\gamma^* = (\sigma^*/lA_m^*)(12/n)^{1/2} \quad (39)$$

where

$$\sigma^* = \frac{A_m^*}{n-2}\left\{\sum_i [\ln A_i - \ln\phi(z_i)]^2\right\}^{1/2} \quad (40)$$

where $\ln\phi(z_i)$ is the value given by (37) at the depth z_i.

For the most part the difference between the starred and unstarred quantities is not significant in this application, and the choice between the normal and the log-normal analyses is somewhat arbitrary. The log-normal assumption was favored in this study because it seemed to account better for the distribution of extreme measured values.

UNCERTAINTIES IN THE DETERMINATION OF $G(\phi)$

We have considered two cases. In the first the only departure of the heat production from the idealized value $\phi(z)$ is that represented by a sine wave of wavelength $\lambda = l/\nu$ where l is the length of the interval of observation (typically the hole depth). If this interval is completely sampled with no observational error, the value of G determined from a regression analysis can contain an error δG as large as

$$\delta G = W(\nu)a_\nu/A_m l \quad (41)$$

where W is given by equation 21b and Figure 3, and a_ν is the amplitude of the perturbation.

In the second case the only departures from the idealized relation $\phi(z)$ are random ones on the hand-sample scale that are assumed to be normally (or log normally) distributed. In this case the standard error γ of G resulting from a linear regression analysis of n specimens spaced equally over the interval l is

$$\gamma = \frac{\sigma}{A_m}\frac{(12/n)^{1/2}}{l} \quad (42)$$

where σ is the standard deviation of the heat-production sample.

As (41) and (42) have similar form, the dependence on l of these two types of errors in the estimate of G can be represented on a single graph (Figure 4). The values of n on the curves refer to the evaluation of γ; the values of ν refer to δG. On the left-hand ordinate scale

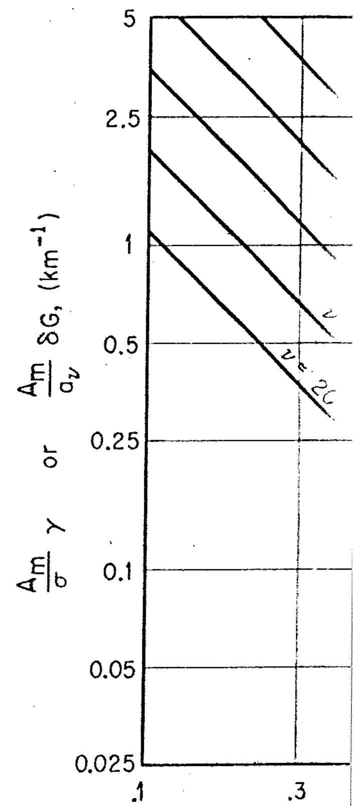

Fig. 4. Relation between ho[...] ated with a large-scale perturb[...] sociated with small-scale rando[...] equally spaced specimens. A_m is [...]

γ is normalized by the ratio of th[...] production to its standard devia[...] by the ratio of the mean heat pro[...] amplitude of the perturbation.

The standard deviation σ of [...] tion samples tends to lie in the ra[...] to 30% of the mean A_m with [...] typical value (Table 1, *Lachenbruc*[...] [1971]). The right-hand ordinate [...] ure 4 gives numerical values of γ w[...] of the ratio substituted. It also gi[...] values of δG for the case in which [...] of the perturbation is 20% of th[...] production. This value is consiste[...] and is probably reasonable to [...] possible magnitudes of δG consist[...] linear heat-flow relation.

...H

$x^* = \sigma^*/(n)^{1/2}$ (38)

$\gamma^* = (\sigma^*/lA_m^*)(12/n)^{1/2}$ (39)

$... \{\sum_i [\ln A_i - \ln \phi(z_i)]^2\}^{1/2}$ (40)

z_i) is the value given by (37) at the

...most part the difference between the ... unstarred quantities is not signifi- ... application, and the choice be- ...ormal and the log-normal analyses ... arbitrary. The log-normal assump- ...red in this study because it seemed ...better for the distribution of ex- ...red values.

...NTIES IN THE DETERMINATION OF $G(\phi)$

...considered two cases. In the first ...arture of the heat production from ... value $\phi(z)$ is that represented by ... of wavelength $\lambda = l/\nu$ where l is ... the interval of observation (typi- ... depth). If this interval is com- ...d with no observational error, the ...etermined from a regression anal- ...in an error δG as large as

$$\delta G = W(\nu) a_\nu / A_m l \quad (41)$$

...ven by equation 21b and Figure ...e amplitude of the perturbation. ...d case the only departures from ...elation $\phi(z)$ are random ones on ...le scale that are assumed to be ...og normally) distributed. In this ...rd error γ of G resulting from a ...n analysis of n specimens spaced ... interval l is

$$= \frac{\sigma}{A_m} \frac{(12/n)^{1/2}}{l} \quad (42)$$

...standard deviation of the heat-...le.

...(42) have similar form, the ... of these two types of errors in ... can be represented on a single ... The values of n on the curves ...uation of γ; the values of ν ... the left-hand ordinate scale

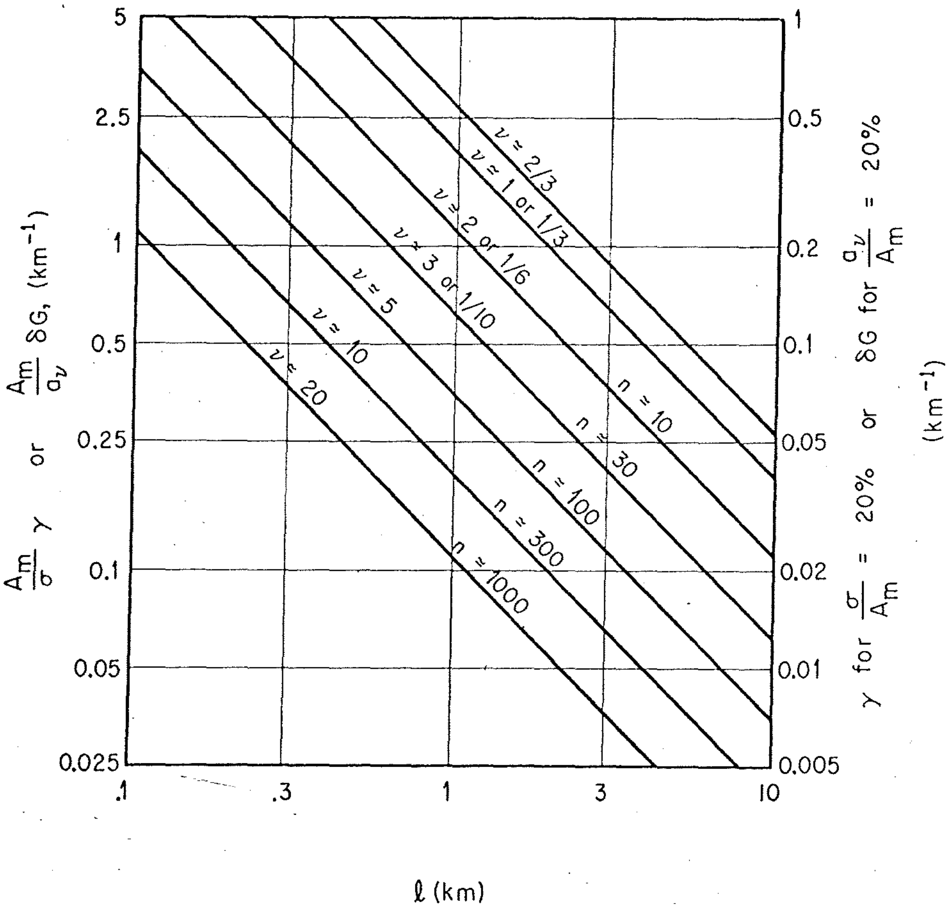

Fig. 4. Relation between hole depth l and the error in the normalized gradient δG associated with a large-scale perturbation of amplitude a_ν and wave number ν; or the error γ associated with small-scale random perturbations with standard deviation σ, sampled with n equally spaced specimens. A_m is the mean heat production.

γ is normalized by the ratio of the mean heat production to its standard deviation, and δG by the ratio of the mean heat production to the amplitude of the perturbation.

The standard deviation σ of heat-production samples tends to lie in the range from 10 to 30% of the mean A_m with 20% being a typical value (Table 1, *Lachenbruch and Bunker* [1971]). The right-hand ordinate scale of Figure 4 gives numerical values of γ with this value of the ratio substituted. It also gives numerical values of δG for the case in which the amplitude of the perturbation is 20% of the mean heat production. This value is consistent with (11) and is probably reasonable to illustrate the possible magnitudes of δG consistent with the linear heat-flow relation.

Table 1 shows the value of $G(\phi)$ that would result from various simple models of ϕ. As these values are of the order of 0.1 km^{-1}, it is clear that uncertainties in G must be kept to a few hundredths km^{-1} if we are to discriminate between the simplest alternative models. Holes drilled for the determination of heat flow and heat production are typically of the order of 0.3 km deep and normally about 10 heat-production samples are taken. Although these seem adequate for the determination of A_m, it is seen (curve $n = 10$, Figure 4) that the standard error γ in G determined from them would generally be several hundred per cent of the larger values of $G(\phi)$ in Table 1. If the number of samples were increased to several hundred, γ could be reduced to less than 0.1 km^{-1} (see curves $n = 300$, $n = 1000$, Figure 4), but this is likely to be a fruitless exercise insofar as the estimate of G is concerned. It is seen from the curve $\nu = 3$ (Figure 4), that no matter how many samples were taken in a 0.3-km hole, a perturbation with a wavelength of 0.1 km and an amplitude of only 20% A_m could cause an error δG in $G(\phi)$ of 0.4 km^{-1}. The curve $\nu = 3$ also represents the case $\nu = 0.1$, which means that a perturbation with a wavelength of 3 km and the same amplitude could cause an error of similar magnitude. Intermediate wavelengths (0.1 to 3 km) could cause substantially larger errors in G, and none of these perturbations would seriously upset the linear heat-flow relation (see equations 7 and 8). Although the higher frequency perturbations might be filtered out by a numerical procedure, the larger wavelengths could not. It is seen that in the worst case ($\nu \simeq 2/3$) a perturbation with a wavelength between 0.4 and 0.5 km could cause and error δG of 0.1 km^{-1} in a 0.3-km hole even if its amplitude were only 1 or 2% of A_m.

A most comprehensive study of the vertical distribution of heat-producing elements has been made in the Conway granite with hundreds of samples from a maximum depth of 0.3 km [*Rogers et al.*, 1965]. According to the foregoing discussion these observations cannot be expected to yield useful estimates of G, although they are, of course, useful for other purposes.

A few holes have been drilled to depths of the order of 3 km in crystalline rock. It is seen (Figure 4) that 100 heat-production samples from such holes would yield $\gamma \sim 0.02$ km^{-1}, and they might lead to useful estimates of $G(\phi)$ if larger wavelength perturbations were not severe. Perturbations with wavelengths of a few tenths of a kilometer, so important in the previous cases, are generally insignificant in a 3-km hole (see curve $\nu = 10$, Figure 4). It is seen, however, that if the phase is unfavorable, wavelengths of the order of the hole depth could cause errors in G of the order of 0.1 km^{-1} even if their amplitude were only 10–20% of A_m. Thus, even in a 3-km hole, departures from the idealized distribution that would not significantly alter the linear heat-flow relation can completely mask those values of $G(\phi)$ predicted by the simple models (Table 1).

SUMMARY

Any member of a large family of vertical distributions of heat production is compatible with the linear heat-flow relation, but the relation is most simply explained by assuming a geochemical tendency toward one particular idealized form $\phi(z)$, unspecified, but the same (except for the value of D) from one province to another. Some arguments have been advanced for assuming this distribution to be a step function, others for its being an exponential function. For these cases the normalized gradient of heat production $[G(\phi) = -\phi^{-1}(d\phi/dz)]$ would be zero, or $D^{-1} \sim 0.1$ km^{-1}, respectively. To distinguish between these (or other) models from measurements of heat-production gradients in boreholes, it is necessary to determine G with an uncertainty substantially less than 0.1 km^{-1}. However, the actual heat production $A(z)$ can depart considerably from the idealized form $\phi(z)$ without measurably affecting the validity of the linear heat-flow relation. In general, such departures can completely obscure $G(\phi)$ in holes to 3 or more km, but the likelihood of their doing so diminishes sharply with increasing hole depth.

In a hole a few hundred meters deep, such as those usually drilled for heat-flow studies, random fluctuations of heat production known to occur on the hand-sample scale generally obviate meaningful estimates, even of the sign, of $G(\phi)$. Even in holes 1 km deep, small-scale fluctuations, combined with perturbations with wavelengths of the order of a kilometer with amplitudes of only a few per cent, will render meaningless most estimates of $G(\phi)$. In 3-km holes the uncertainty in $G(\phi)$ due to small-scale random perturbation is reduced to acceptable levels with 100 or so samples. However, moderate departures from ϕ with wavelengths greater than 1 or 2 km can create errors in the estimate of the same order as the quantity sought if the phase is unfavorable.

Although estimates of $G(\phi)$ from measurements in individual holes might be highly uncertain, several determinations in separate holes might be collectively significant if each hole could be assumed to represent the same idealized gradient, $G(\phi)$, and departures of all wavelengths from it were random.

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