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A simple method of calculating climatic corrections to heat flow measurements

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A series approximation for calculating climatic corrections to include the effect of all of the Pleistocene glaciations is derived. The method is useful for boreholes less than 1200 m deep, and only two terms of the series are required for depths less than 600 m. About 70% of the effect at the surface is a result of the most recent deglaciation and this percentage increases with depth.

The theory of the effect of climatic changes on thermal gradients was first given by Lane (1923). The general method, as described in great detail by Birch (1948), approximates the climatic fluctuation by a series of short intervals of constant temperature. The effect of such a temperature variation on the surface of a semi-infinite medium of constant thermal diffusivity, s, is given as follows, after Ingersoll *et al.* (1954 p. 121).

$$[1] \quad \Delta T(z) = \sum_{i=1}^{\infty} T_i \left[\operatorname{erf} \frac{z}{2\sqrt{st_i}} - \operatorname{erf} \frac{z}{2\sqrt{st_{i-1}}} \right]$$

where z is the depth, s is the thermal diffusivity, T_i is the constant temperature during the time interval $t_{i-1} < t < t_i$ (time before present is positive) and

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du$$

If, further, it may be assumed that the temperature for all time before a given time, t_N, was a constant, A, then Equation [1] becomes a finite sum

$$[2] \quad \Delta T(z) = \sum_{i=1}^{N} C_i \left[\operatorname{erf} \frac{z}{2\sqrt{st_i}} - \operatorname{erf} \frac{z}{2\sqrt{st_{i-1}}} \right]$$

where $\qquad C_i = T_i - A$

With the use of a digital computer, the time intervals may be made sufficiently small (N large) that any desired degree of accuracy may be obtained, no matter how complicated the climatic variation.

Because of the possible significance of climatic corrections (Crain 1968), it is desirable to be able to calculate a good estimate of the climatic corrections imply and rapidly, without resorting to computer programming. It was shown by Crain (1967) that the small details of the climatic variations have little ultimate effect on the heat flow correction. The critical factors are the times of the transitions between the glacial and interglacial periods, and the difference in mean annual temperature between the glacial and interglacial periods. Thus the assumption of constant temperature during each of the glacial periods does not contribute significant error. As an additional simplification, paleoclimatic results indicate that mean temperatures during the glacial periods were roughly equal, as were interglacial temperatures (Emiliani 1961).

Many workers have assumed that the most recent deglaciation only need be considered for a climatic correction. The extent to which this assumption is true will be examined later.

If one uses the justified approximation of constant temperature throughout each of the glacial periods, Equation [2] reduces to

$$[3] \quad \Delta T(z) = T_0 \sum_{i=1}^{N} (-)^i \operatorname{erf} \frac{z}{2\sqrt{st_i}}$$

where T_0 is the difference between glacial and interglacial temperature, and t_i are the times of the transitions from glacial to interglacial and from interglacial to glacial.

A series expansion for the error function which may be derived using the definition of erf(x) and the Taylor series for the exponential is as follows:

$$[4] \quad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{1!3} + \frac{x^5}{2!5} - \frac{x^7}{3!7} + \cdots \right]$$

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Combining Equations [3] and [4] gives

$$[5] \quad \Delta T(z) = \frac{2T_0}{\sqrt{\pi}} \sum^{N} (-)^i \left[\frac{z}{2q_i} - \frac{z^3}{1!3 \cdot 2^3 q_i^3} + \frac{z^5}{2!5 \cdot 2^5 q_i^5} - \cdots \right]$$

where $q_i = \sqrt{st_i}$.

If [5] is rearranged by collecting like powers of z, and is then differentiated, the change in heat flow caused by the climatic effect is given.

$$[6] \quad \Delta Q(z) = k \frac{\partial \Delta T(z)}{\partial z} = \frac{kT_0}{\sqrt{\pi}} \left[\sum_{i=1}^{N} \frac{(-)^i}{q_i} - \frac{z^2}{1!2^2} \times \sum_{i=1}^{N} \frac{(-)^i}{q_i^3} + \frac{z^4}{2!2^4} \sum_{i=1}^{N} \frac{(-)^i}{q_i^5} - \cdots \right]$$

where k is the thermal conductivity (by convention upward heat flow is positive).

Equation [6] can be rewritten as

$$[7] \quad \Delta Q(z) = \frac{kT_0}{\sqrt{\pi}} \left[\frac{1}{q_1} \sum_{i=1}^{N} (-)^i p_i - \frac{z^2}{1!2^2 q_1^3} \times \sum_{i=1}^{N} (-)^i p_i^3 + \frac{z^4}{2!2^4 q_1^5} \sum_{i=1}^{N} (-)^i p_i^5 - \cdots \right]$$

where

$$p_i = \frac{q_1}{q_i} = \sqrt{\frac{t_1}{t_i}}$$

This series converges for all finite z, but converges rapidly only for small values. Let us then examine the series to determine how many terms are necessary for calculation. Consider the expression, P_m, given by

$$[8] \quad P_m = \sum_{i=1}^{N} (-)^i p_i^m \qquad m = 1, 2, 3 \cdots$$

The magnitude of this coefficient is less than 1 for $m \geq 1$, since $t_i \geq t_{i-1}$. Thus to examine convergence one may look at the series

$$[9] \quad S(z) = \frac{kT_0}{\sqrt{\pi}} \left[\frac{1}{q_1} - \frac{z^2}{1!2^2 q_1^3} + \frac{z^4}{2!2^4 q_1^5} - \cdots \right]$$

Since this is a monotonically decreasing alternating series (Wylie 1953, p. 417), the error caused by truncating this series is less than the first term dropped. One may then substitute the following typical values of the parameters to study the behavior of the series. The time of the most recent deglaciation, t_1, is generally placed in the vicinity of 10^4 years ($\sim 3 \times 10^{11}$ s), a typical diffusivity is 10^{-2} cm^2 s^{-1}, and an average conductivity is about 5×10^{-3} cal °C cm^{-1} s^{-1}. There is evidence from paleotemperature analysis and theoretical glaciology, that T_0 is about 10 °C (Emiliani 1961; Terasmae 1961; Robin 1955; Crain 1967). If one assumes a heat flow correction of 1×10^{-8} cal cm^{-2} s^{-1} to be insignificant, then the first term of Equation [9] will be accurate for depths given by

$$[10] \quad \frac{kT_0 z^2}{\sqrt{\pi} 1!2^2 (st_1)^{3/2}} \leq 10^{-8}$$

that is $z \leq 150$ m.

The first two terms of the series will suffice for depths for which the third term is negligible, namely

$$[11] \quad \frac{kT_0 z^4}{\sqrt{\pi} 2!2^4 (st_1)^{5/2}} \leq 10^{-8}$$

that is $z \leq 600$ m.

Similarly it may be shown that the first three terms only are required for depths less than 1200 m.

It should be noted that the depth ranges quoted refer to an "average" diffusivity of 10^{-2} c.g.s. units. These depths will decrease for extremely low diffusivities and correspondingly increase for high values of this parameter. The fact that fewer terms are required to correct a shallow borehole is purely a mathematical phenomenon. It is still true, of course, that results from deeper boreholes should be viewed with more confidence, since the absolute value of the correction decreases rapidly with depth.

Let us now examine P_1 for typical values of the t_i's. A value of P_1 near -1 would indicate that the assumption that the most recent deglaciation alone is significant, is valid. From the paleotemperature analysis of Emiliani (1961) the dates and resulting p_i's for the glacial–interglacial transitions are shown in Table I. This gives a value of P_1 of -0.66, while $P_3 = -0.95$, and $P_m = -1.00$ for $m > 3$.

TABLE I

Values of the p_i's

i	t_i (years)	P_i	p_i^3	p_i^5	p_i^7
1	11 000	1.0000	1.0000	1.0000	1.0000
2	65 000	0.4114	0.0696	0.0118	0.0020
3	98 000	0.3350	0.0376	0.0042	0.0005
4	130 000	0.2909	0.0246	0.0021	0.0002
5	175 000	0.2507	0.0158	0.0010	0.0001
6	205 000	0.2316	0.0124	0.0007	0.0000
7	265 000	0.2037	0.0085	0.0004	0.0000
8	300 000	0.1915	0.0070	0.0003	0.0000

$P_m = \sum_{i=1}^{8} (-)^i p_i^m \quad P_1 = -0.6640 \quad P_3 = -0.9483 \quad P_5 = -0.9907 \quad P_7 = -0.9984$

Thus for this particular sample calculation the climatic correction is given by

$$[12] \quad \Delta Q(z) = -0.34 + 0.37\left(\frac{z}{10^3}\right)^2 - 0.16\left(\frac{z}{10^3}\right)^4$$

where Q is in units of μcal $\text{cm}^{-2}\ \text{s}^{-1}$, and z is in meters.

The corresponding equation which ignores the interglacial periods is

$$[13] \quad \Delta Q(z) = -0.49 + 0.39\left(\frac{z}{10^3}\right)^2 - 0.16\left(\frac{z}{10^3}\right)^4$$

The minus sign indicates that the climatic effect is in opposition to the normal gradient. It can be seen that the difference between the two expressions is about 0.15 μcal $\text{cm}^{-2}\ \text{s}^{-1}$ at the surface and decreases as the depth increases.

Jessop (1968) has calculated approximate climatic corrections for some Canadian boreholes using an extension of the "near surface approximation" of Jaeger (1965), that is, the first term of the Equation [7], and considered only the onset and end of the Wisconsin glaciation, that is, P_1 and P_3. This was an improvement over earlier attempts, including the author's (Crain 1967), which considered only the most recent deglaciation or which greatly underestimated the amplitude of the climatic fluctuations (Birch 1948). On the other hand, it is only a good estimate of the correction for shallow boreholes, while the series solution of this paper extends the useful range of the calculation to all depths of practical interest.

The following problems are inherent in any method of climatic correction:

(1) The chronology of the Pleistocene glaciations is not accurately known. Fortunately, the most recent climatic changes are dated with the most confidence and are the most critical for the calculation of climatic corrections. In particular, the mean temperature before the Pleistocene fluctuations is not known accurately, so that the base line to which all measurements should be corrected is uncertain. If this average temperature differs greatly from the present mean, a small "zero-order term" caused by this effect is possible.

(2) The value of T_0 is relatively unknown, although there is some evidence, previously mentioned, which suggests that 10 °C is not unreasonable.

(3) The assumption of constant diffusivity may contribute significant errors where large diffusivity contrasts exist.

The following are the main conclusions of this note:

(1) Equation [7] is an approximate expression for the calculation of climatic corrections including the effects of all glacial periods. The series is sufficiently accurate when only a few terms are considered and thus calculation is simple and rapid. Slide-rule accuracy is sufficient.

(2) The most recent deglaciation accounts for at least 70% of the climatic correction, and this contribution increases with the depth of the borehole.

(3) The constant and quadratic terms of Equation [7] are sufficient to calculate an accurate climatic correction to depths of about 600 m.

(4) The use of a "near surface approximation" should be restricted to depths of less than 150 m.

(5) The use of this series method is not recommended for hole-depths greater than 1200 m.

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