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Harmonic Temperature Waves in a Horizontally Layered Medium

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Abstract. A recurrence formula is given for determining the attenuation and phase change of a harmonic temperature wave in a horizontally layered medium. Some diagrams illustrate the results in the case of a two-layer medium. They can be used for geothermal mapping of an aquifer.

Key words: Geothermics – Temperature Waves.

Introduction

In the investigation of the penetration of temperature variations into the earth, it is of interest to know the dependency of the attenuation and phase change of harmonic temperature waves in a horizontally layered medium as a function of depth, frequency, the thermal constants, and the depths of the layer boundaries. One example of this type of problem is that of detecting flat, water-bearing layers using geothermic methods (Cartwright, 1968). The solution of the general problem is given here together with the results of certain cases in diagrammatic form.

Theory

In every layer j ($j=1,2,\ldots,n$) of the n-fold stratified medium with thermal conductivity λ_j and thermal diffusivity $a_j=\lambda_j/\varrho_j c_j$ (ϱ_j = density, c_j = specific heat), the temperature T_j, a function of time t and depth z must satisfy the equation of heat conduction:

$$\frac{\partial T_j}{\partial t} = a_j \cdot \frac{\partial^2 T_j}{\partial z^2} \quad \begin{array}{l}\text{for } z_{j-1} \le z \le z_j \\ \text{and } z_0 = 0,\ z_n \to \infty\end{array}, \tag{1}$$

where z_j ($j=1,2,\ldots,n-1$) represents the boundaries of the layers. For every internal layer boundary the internal boundary conditions

$$T_j = T_{j+1} \tag{2}$$

$$\lambda_j \cdot \frac{\partial T_j}{\partial z} = \lambda_{j+1} \cdot \frac{\partial T_{j+1}}{\partial z} \quad \text{for } z = z_j\ (j=1,2,\ldots,n-1) \tag{3}$$

must be satisfied, and for the last layer

$$T_n \to 0 \quad \text{as } z \to \infty \tag{4}$$

where a constant additive temperature and a linear gradient have been omitted.

At the earth's surface $z=0$ the boundary condition is

$$\alpha \cdot \frac{\partial T_1}{\delta z} + \beta \cdot T_1 = \gamma \cdot \cos(\omega t - \varepsilon) = \gamma \cdot \mathrm{Re}\,\{\exp(i(\omega t - \varepsilon))\} \tag{5}$$

where α, β, γ are given real constants . $\omega = 2\,\pi/\tau$ is the angular frequency (τ = period), ε is the initial phase shift of the temperature wave in the atmosphere, Re denotes the real part of the relevant function, and i is the imaginary unit. Condition (5), which describes the heat transfer between the earth's surface and the atmosphere ($z<0$), includes the particular condition for a given harmonic temperature with amplitude T_0

$$T_1 = T_0 \cdot \cos(\omega t - \varepsilon) \quad \text{for } z = 0 \tag{5a}$$

($\alpha=0$, $\beta=1$, $\gamma=T_0$), and also the case of a given heat flow density

$$q = -\lambda_1 \frac{\partial T_1}{\delta z} = Q \cdot \cos(\omega t - \varepsilon) \quad \text{for } z = 0 \tag{5b}$$

($\alpha = -\lambda_1$, $\beta=0$, $\gamma=Q$).

The general solution of the equation of heat conduction can be found by separation of variables:

$$T_j(z,t) = \mathrm{Re}\{C \cdot [A_j \cdot \exp(-p_j z) + B_j \cdot \exp(p_j z)] \cdot \exp(i(\omega t - \varepsilon))\} \tag{6}$$

where $p_j = (i\omega/a_j)^{1/2}$, $\mathrm{Re}\{p_j\} \geq 0$.

C, A_j, B_j are complex constants which must be determined from conditions (2)–(5). From conditions (2) and (3) the following relationships can be derived from the general solution (6) by solving the resulting equations with respect to A_j and B_j:

$$\begin{aligned} A_j &= \frac{\lambda_j \cdot p_j + \lambda_{j+1} \cdot p_{j+1}}{2\lambda_j p_j} \cdot \exp((p_j - p_{j+1}) \cdot z_j) \cdot A_{j+1} + \\ &\quad \frac{\lambda_j \cdot p_j - \lambda_{j+1} \cdot p_{j+1}}{2\lambda_j p_j} \cdot \exp((p_j + p_{j+1}) \cdot z_j) \cdot B_{j+1} \\ B_j &= \frac{\lambda_j \cdot p_j - \lambda_{j+1} \cdot p_{j+1}}{2\lambda_j \cdot p_j} \cdot \exp(-(p_j + p_{j+1}) \cdot z_j) \cdot A_{j+1} + \\ &\quad \frac{\lambda_j \cdot p_i + \lambda_{j+1} \cdot p_{j+1}}{2\lambda_j \cdot p_j} \cdot \exp(-(p_j - p_{j+1}) \cdot z_j) \cdot B_{j+1} \\ &\qquad (j = n-1, n-2, \ldots, 1) \end{aligned} \tag{7}$$

From (4), B_n must be zero. Setting A_n arbitrarily equal to 1, then (7) is a system of recurrence formulae which yield A_1, B_1. From condition (5), one can then obtain the normalizing constant

$$C = \frac{\gamma}{\beta(A_1 + B_1) - \alpha p_1 (A_1 - B_1)}, \tag{8}$$

thereby completely solving the problem.

If one designates, for the special case of the boundary condition

$$T_1 = \cos(\omega t - \varepsilon) = \mathrm{Re}\,\{\exp(i(\omega t - \varepsilon))\} \quad \text{for } z = 0 \tag{5c}$$

i.e. $\alpha=0$, $\beta=\gamma=1$, the normalizing constant as

$$C^* = 1/(A_1 + B_1) \tag{8a}$$

then it can be seen that for a given sequence of layers with given thermal constants, it is sufficient to solve the special case of the boundary condition (5c). The solution for the general boundary condition (5) is then obtained by multiplying the former solution by a factor

$$V = \frac{C}{C^*} = \frac{\gamma(A_1 + B_1)}{\beta(A_1 + B_1) - \alpha p_1 (A_1 - B_1)}. \tag{9}$$

In other words: the attenuation and phase change of the wave within the medium is independent of the boundary condition at the surface of the medium. The boundary condition (5) simply introduces an additional constant reduction in amplitude and an additional phase shift at the surface of the medium.

Numerical Results

In order to reduce the number of variables for a numerical solution it is advisable to introduce dimensionless variables. That depth of penetration

$$d = (2\,a_1/\omega)^{1/2} = (a_1 \cdot \tau/\pi)^{1/2} = 2^{1/2}/|p_1| \tag{10}$$

of a harmonic wave in an unlayered medium with thermal diffusivity a_1 for which the amplitude has decayed to the e-th part, can serve as a reference value. The variables p_j and z are transformed as follows:

$$p_j \to \tilde{p}_j = p_j \cdot d, \quad z \to \tilde{z} = z/d. \tag{11}$$

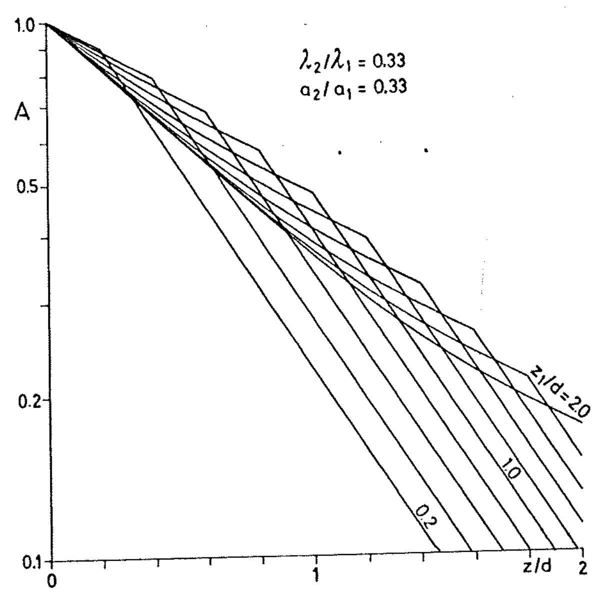

Fig. 1a

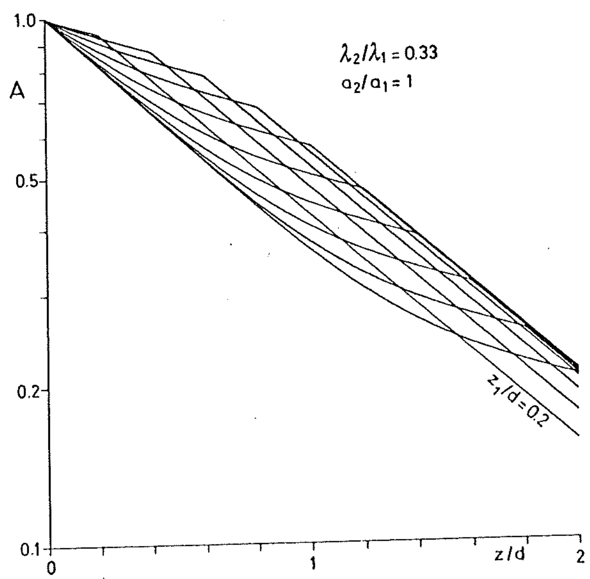

Fig. 1b

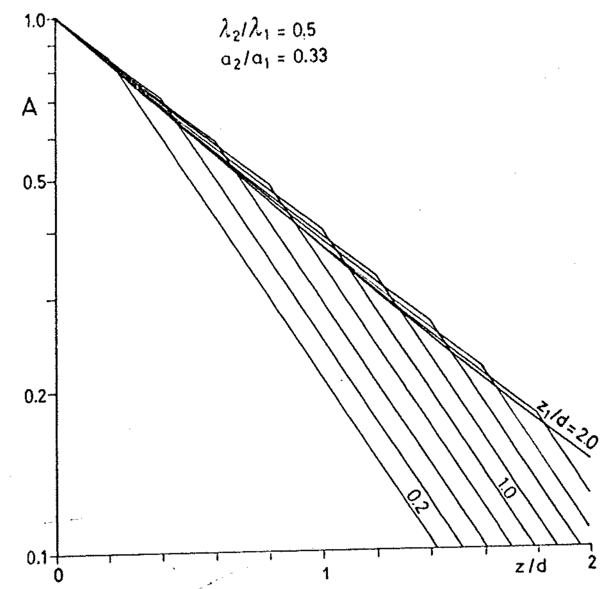

Fig. 1c

Fig. 1a–c. Amplitude A of a harmonic temperature wave as a function of the relative depth z/d and the relative depth of the boundary z_1/d (at intervals of 0.2) for three different two-layer cases

Figs. 1 and 2 show the amplitude relationship and the phase change as a function of the relative depth z/d and the relative depth of the boundary layer z_1/d for the two-layer case for three combinations of the ratio of thermal diffusivities ($a_2/a_1 = 0.33$,1, and 0.33), and of the ratio of thermal conductivities ($\lambda_2/\lambda_1 = 0.33$, 0.33 and 0.5).

In case of Figs. 1c and 2c ($a_2/a_1 = 0.33$, $\lambda_2/\lambda_1 = 0.5$), it was possible to make a direct comparison with results which Kappelmeyer and Hänel (1974) obtained using a method of finite differences. Their results, however, are given in comparison with a temperature distribution in an homogeneous half-space with thermal diffusivity a_1. For that case the amplitude relationship and phase change are given by

$$A^* = \exp(-z/d), \varphi^* = -(180/\pi) \cdot z/d. \tag{12}$$

A particular application is the geothermal mapping of a water-bearing layer beneath a relatively dry overburden (aquifer), as carried out by Cartwright (1968). It depends on measuring the temperature difference between a two-layer case (aquifer present) and a one-layer case (no aquifer) as a function of the measuring depth and the season of the year, where the yearly temperature variations are used in the investigation.

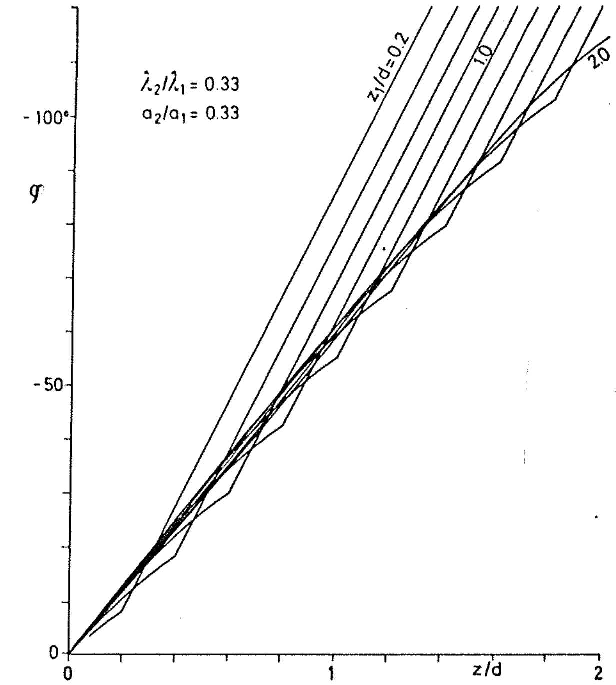

Fig. 2a

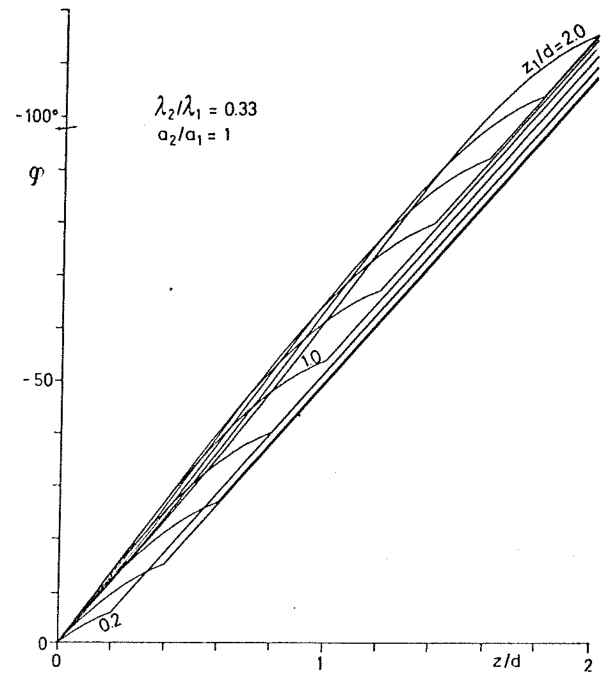

Fig. 2b

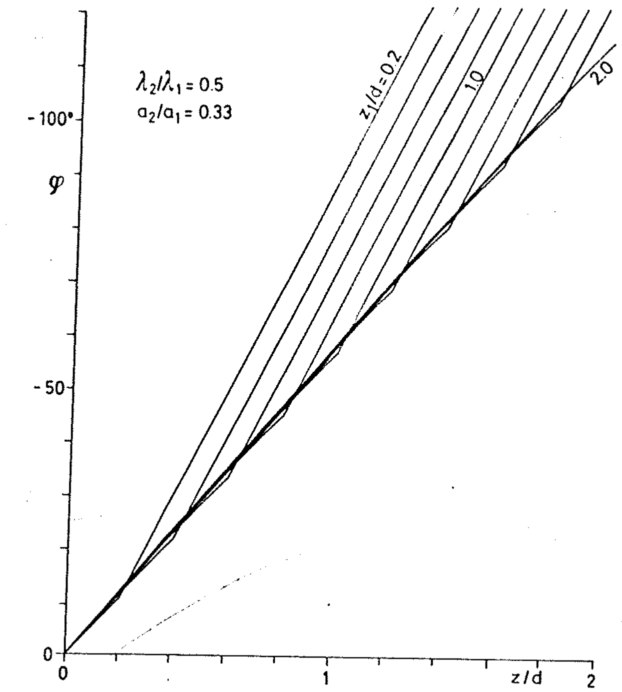

Fig. 2c

Fig. 2a–c. Phase shift φ of a harmonic temperature wave. Otherwise as Fig. 1

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