# Multivariable Curve Interpolation

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*Abstract.* The problem of defining a smooth surface through an array of points in space is well known. Several methods of solution have been proposed. Generally, these restrict the set of points to be one-to-one defined over a planar rectangular grid ($X,Y$-plane). Then a set of functions $Z = F(X,Y)$ is determined, each of which represents a surface segment of the composite smooth surface. In this paper, these ideas are generalized to include a much broader class of permissible point array distributions: namely (1) the point arrangement (ordering) is topologically equivalent to a planar rectangular grid, (2) the resulting solution is a smooth composite of parametric surface segments, i.e. each surface piece is represented by a vector (point)-valued function. The solution here presented is readily applicable to a variety of problems, such as closed surface body definitions and pressure envelope surface definitions. The technique has been used successfully in these areas and others, such as numerical control milling, Newtonian impact and boundary layer.

## *Problem Description*

Let $\mathbf{P}_{i,j}$ be an array of $m \cdot n$ distinct points in space, $i=0, \cdots, m-1$, $j=0, \cdots, n-1$, arranged so that the structure obtained by connecting adjacent points by straight-line segments is topologically equivalent to an $m \times n$ planar rectangular grid (see Figure 1). It is required to construct a smooth surface, $\Gamma$, which passes exactly through the $\mathbf{P}_{i,j}$. The computations of the resulting structure, $\Gamma$, will be adapted to high speed digital techniques, so that the definition of $\Gamma$ must not be such that applications yield calculations of a high degree of complexity.

## *Basic Curve Segment*

Let $\mathbf{A}$, $\mathbf{B}$ be two points in space and let $\mathbf{T_A}$, $\mathbf{T_B}$ be two direction vectors defined at $\mathbf{A}$, $\mathbf{B}$ respectively. We construct a space curve $K$ through $\mathbf{A},\mathbf{B}$ tangent to $\mathbf{T_A}$, $\mathbf{T_B}$ which has the following form:

$$\mathbf{P}(u) = \sum_{i=0}^{3} \mathbf{R}_i u^i, \quad \text{for} \quad 0 \leqq u \leqq 1, \quad \mathbf{P}(u) \in K, \tag{1}$$

such that $\mathbf{P}(0) = \mathbf{A}$, $\mathbf{P}(1) = \mathbf{B}$, $\left.\dfrac{d\mathbf{P}}{du}\right|_{u=0} = \mathbf{T_A}$, and $\left.\dfrac{d\mathbf{P}}{du}\right|_{u=1} = \mathbf{T_B}$.

Substituting these conditions in (1) and solving the resulting system of equations for $\mathbf{R}_0$, $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$ gives

$$\mathbf{P}(u) = u^3[2(\mathbf{A} - \mathbf{B}) + \mathbf{T_A} + \mathbf{T_B}] + u^2[3(\mathbf{B} - \mathbf{A}) - 2\mathbf{T_A} - \mathbf{T_B}] + u\mathbf{T_A} + \mathbf{A}. \tag{2}$$

The notation $K = (\mathbf{A}, \mathbf{B}, \mathbf{T_A}, \mathbf{T_B})$ is used.

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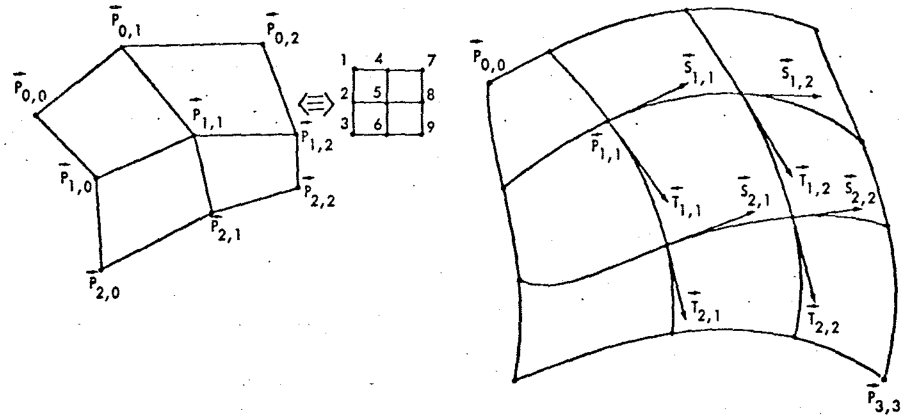


FIGS. 1–2

*Composite Curve Construction*

Consider two families of composite curves, one passing through (for each $i = 0, \cdots, m-1$) $\mathbf{P}_{i,j}$, $j=0, \cdots, n-1$, and the other passing through (for each $j=0, \cdots, n-1$,) $\mathbf{P}_{i,j}$, $i=0, \cdots, m-1$ (see Figure 2). Extract from these curves tangent directions $\mathbf{S}_{i,j}$ and $\mathbf{T}_{i,j}$ at the $\mathbf{P}_{i,j}$, where $\mathbf{S}_{i,j}$ is tangent to the composite curve of increasing $j$-value and $\mathbf{T}_{i,j}$ is tangent to the composite curve of increasing $i$-value. Each composite curve is a collection of basic curve segments joining successive points of it such that tangency is preserved at their junctions. Since each basic curve section of a composite curve is defined by two points and two tangents, it is necessary to devise a means for defining tangents at the points through which the composite curve must pass. Therefore, consider in general a set of $p$ distinct points $\mathbf{P}_k$, $k = 0, \cdots, p-1$, in space and let $\mathbf{Y}_k$ denote the tangent sought at $\mathbf{P}_k$. Specifying second derivative vectors equal at the points $\mathbf{P}_k$, $k = 1, \cdots, p-2$, gives us the following recursive relation

$$\mathbf{Y}_k + 4\mathbf{Y}_{k+1} + \mathbf{Y}_{k+2} = 3(\mathbf{P}_{k+2} - \mathbf{P}_k) \quad k = 0, \cdots, p-3,$$

representing $p-2$ equations in $p$ unknowns. One may now assume that $\mathbf{Y}_0$ and $\mathbf{Y}_{p-1}$ are known, thus completing the solution.

*Basic Surface Segment*

Now assign to each $i$, $j$ the four basic curve segments $K_1 = (\mathbf{P}_{i,j}, \mathbf{P}_{i+1,j}, \mathbf{T}_{i,j}\mathbf{T}_{i+1,j})$, $K_2$, $K_3$, and $K_4$ (see Figure 3). Then the basic surface segment $\mathfrak{S}_{i,j}$ is constructed as follows: parametrize $K_1$, $K_2$ with respect to the variable $u$, and parametrize $K_3$, $K_4$ with respect to the variable $v$. Then locate the points $\mathbf{P}(u) \in K_1$ and $\mathbf{Q}(u) \in K_2$. Now interpolate from $K_3$ to $K_4$, at the value $u$, by defining a basic curve segment with $\mathbf{P}(u)$, $\mathbf{Q}(u)$ as beginning and end points respectively. This intermediate curve segment is parametrized with respect to the variable $v$ and is of the form $(\mathbf{P}(u), \mathbf{Q}(u), \mathbf{X}(u), \mathbf{Y}(u))$, where

$$\begin{aligned} \mathbf{X}(u) &= (au^3 + bu^2 + cu + d)\mathbf{S}_{i,j} + (eu^3 + fu^2 + gu + h)\mathbf{S}_{i+1,j}, \\ \mathbf{Y}(u) &= (au^3 + bu^2 + cu + d)\mathbf{S}_{i,j+1} + (eu^3 + fu^2 + gu + h)\mathbf{S}_{i+1,j+1}. \end{aligned} \tag{3}$$

We wish to determine the coefficients in (3) such that tangency is preserved across the borders $K_3$ and $K_4$. Note that by definition of $\mathbf{X}$ and $\mathbf{Y}$ there is already tangency across $K_1$, $K_2$, since $\mathbf{X}(u)$ is calculated for $\mathcal{S}_{i,j}$ as $\mathbf{Y}(u)$ is calculated for $\mathcal{S}_{i,j-1}$, $j \neq 0$. (Assume now that $i \leqq m-3$, i.e. $\mathcal{S}_{i+1,j}$ exists.) The following conditions must therefore be satisfied:

$$\begin{aligned}
&\text{(i)} && \mathbf{X}(0) = \mathbf{S}_{i,j}, \qquad \mathbf{X}(1) = \mathbf{S}_{i+1,j} \\
&\text{(ii)} && \mathbf{Y}(0) = \mathbf{S}_{i,j+1}, \qquad \mathbf{Y}(1) = \mathbf{S}_{i+1,j+1} \\
&\text{(iii)} && \left.\frac{\partial \mathbf{P}(u,v)}{\partial u}\right|_{u=1} = \left.\frac{\partial \mathbf{Q}(u,v)}{\partial u}\right|_{u=0},
\end{aligned} \tag{4}$$

where $\mathbf{P}(u, v) \in \mathcal{S}_{i,j}$, $\mathbf{Q}(u, v) \in \mathcal{S}_{i+1,j}$, and

$$\mathbf{P}(u, v) = \sum_{p=0}^{3} \sum_{q=0}^{3} u^p v^q \mathbf{R}_{p,q} \qquad 0 \leqq u, v \leqq 1. \tag{5}$$

The $\mathbf{R}_{p,q}$ are determined from the above definitions of $\mathbf{P}(u)$, $\mathbf{Q}(u)$, $\mathbf{X}(u)$ and $\mathbf{Y}(u)$, and in particular depend on $a, b, c, d, e, f, g, h$; a similar relation holds for $\mathbf{Q}(u, v)$. (See Figure 4.) Now, from (iii) of (4) above, equate coefficients of like powers in $v$, obtaining

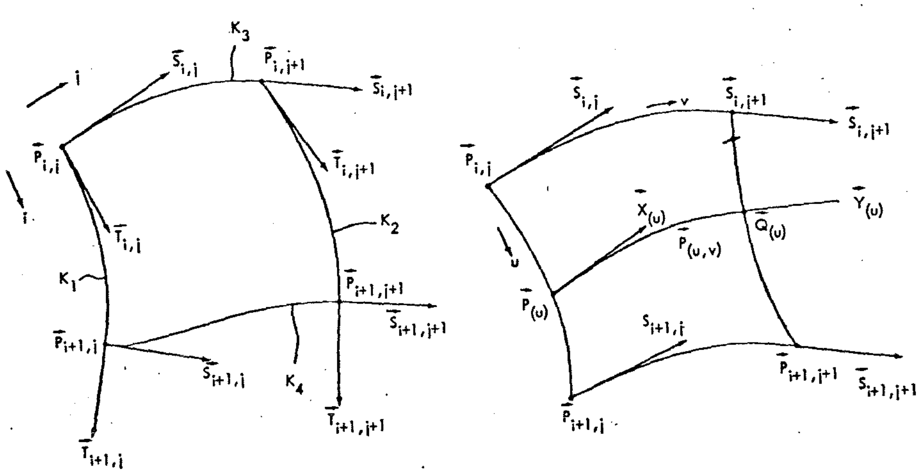
$$c + 2b + 3a = 0, \qquad g + 2f + 3e = c, \qquad g = 0 \tag{6}$$

and from (i), (ii) above equate coefficients of like terms to obtain

$$d = 1, \quad h = 0, \quad a + b + c + d = 0, \quad e + f + g + h = 1. \tag{7}$$

Assembling 6 and 7 one finds

$$\begin{aligned}
3a + 2b + c \qquad\qquad &= 0 \\
- c + 3e + 2f &= 0 \\
a + b + c \qquad\qquad &= -1 \\
e + f &= 1.
\end{aligned} \tag{8}$$




FIGS. 3–4

Another relation among $a, b, c, e, f$ follows if we specify that the quantity $(\partial \mathbf{P}(u, v)/\partial u)\mid_{u=0}$ is interpolated similarly to $\mathbf{X}(u)$, i.e.

$$\left.\frac{\partial \mathbf{P}(u, v)}{\partial u}\right|_{u=0} = (av^3 + bv^2 + cv + d)\mathbf{T}_{i,j} + (ev^3 + fv^2 + gv + h)\mathbf{T}_{i,j+1}.$$

Using (5) to expand this partial derivative and comparing it with (8) we find $c = 0$. So, from (8) it follows that

$$a = 2, \qquad b = -3, \qquad e = -2, \qquad f = 3. \tag{9}$$

Note also that $(\partial \mathbf{P}(u, v)/\partial u)\mid_{u=1}$ is similarly interpolated from $\mathbf{T}_{i+1,j}$ to $\mathbf{T}_{i+1,j+1}$. Thus, the equation for $\mathcal{S}_{i,j}$ is (5), where

$$\begin{aligned}
\mathbf{R}_{0,0} &= \mathbf{P}_{i,j}\\
\mathbf{R}_{0,1} &= \mathbf{S}_{i,j}\\
\mathbf{R}_{0,2} &= 3(\mathbf{P}_{i,j+1} - \mathbf{P}_{i,j}) - (2\mathbf{S}_{i,j} + \mathbf{S}_{i,j+1})\\
\mathbf{R}_{0,3} &= 2(\mathbf{P}_{i,j} - \mathbf{P}_{i,j+1}) + (\mathbf{S}_{i,j} + \mathbf{S}_{i,j+1})\\
\mathbf{R}_{1,0} &= \mathbf{T}_{i,j}\\
\mathbf{R}_{1,1} &= 0\\
\mathbf{R}_{1,2} &= 3(\mathbf{T}_{i,j+1} - \mathbf{T}_{i,j})\\
\mathbf{R}_{1,3} &= 2(\mathbf{T}_{i,j} - \mathbf{T}_{i,j+1})\\
\mathbf{R}_{2,0} &= 3(\mathbf{P}_{i+1,j} - \mathbf{P}_{i,j}) - (2\mathbf{T}_{i,j} + \mathbf{T}_{i+1,j})\\
\mathbf{R}_{2,1} &= 3(\mathbf{S}_{i+1,j} - \mathbf{S}_{i,j})\\
\mathbf{R}_{2,2} &= 3[3(\mathbf{P}_{i+1,j+1} - \mathbf{P}_{i,j+1} + \mathbf{P}_{i,j} - \mathbf{P}_{i+1,j}) + 2(\mathbf{T}_{i,j} - \mathbf{T}_{i,j+1}) + (\mathbf{T}_{i+1,j} - \mathbf{T}_{i+1,j+1})\\
&\quad + 2(\mathbf{S}_{i,j} - \mathbf{S}_{i+1,j}) + (\mathbf{S}_{i,j+1} - \mathbf{S}_{i+1,j+1})]\\
\mathbf{R}_{2,3} &= 2[3(\mathbf{P}_{i+1,j} - \mathbf{P}_{i,j} + \mathbf{P}_{i,j+1} - \mathbf{P}_{i+1,j+1}) + 2(\mathbf{T}_{i,j+1} - \mathbf{T}_{i,j}) + (\mathbf{T}_{i+1,j+1} - \mathbf{T}_{i+1,j})]\\
&\quad + 3(\mathbf{S}_{i+1,j} + \mathbf{S}_{i+1,j+1} - \mathbf{S}_{i,j} - \mathbf{S}_{i,j+1})\\
\mathbf{R}_{3,0} &= 2(\mathbf{P}_{i,j} - \mathbf{P}_{i+1,j}) + \mathbf{T}_{i,j} + \mathbf{T}_{i+1,j}\\
\mathbf{R}_{3,1} &= 2(\mathbf{S}_{i,j} - \mathbf{S}_{i+1,j})\\
\mathbf{R}_{3,2} &= 3[2(\mathbf{P}_{i,j+1} - \mathbf{P}_{i+1,j+1} + \mathbf{P}_{i+1,j} - \mathbf{P}_{i,j}) + (\mathbf{T}_{i,j+1} + \mathbf{T}_{i+1,j+1}) - (\mathbf{T}_{i,j} + \mathbf{T}_{i+1,j})]\\
&\quad + 4(\mathbf{S}_{i+1,j} - \mathbf{S}_{i,j}) + 2(\mathbf{S}_{i+1,j+1} - \mathbf{S}_{i,j+1})\\
\mathbf{R}_{3,3} &= 2[2(\mathbf{P}_{i,j} - \mathbf{P}_{i+1,j} + \mathbf{P}_{i+1,j+1} - \mathbf{P}_{i,j+1}) + (\mathbf{T}_{i,j} + \mathbf{T}_{i+1,j}) - (\mathbf{T}_{i,j+1} + \mathbf{T}_{i+1,j+1})\\
&\quad + (\mathbf{S}_{i,j} - \mathbf{S}_{i+1,j} + \mathbf{S}_{i,j+1} - \mathbf{S}_{i+1,j+1})]
\end{aligned}$$

We use the notation

$$\mathcal{S}_{i,j} = (\mathbf{P}_{i,j}, \mathbf{P}_{i+1,j}, \mathbf{P}_{i,j+1}, \mathbf{P}_{i+1,j+1}, \mathbf{T}_{i,j}, \mathbf{T}_{i+1,j}, \mathbf{T}_{i,j+1}, \mathbf{T}_{i+1,j+1}, \mathbf{S}_{i,j}, \mathbf{S}_{i+1,j}, \mathbf{S}_{i,j+1}, \mathbf{S}_{i+1,j+1}).$$

## *An Interesting Property*

We consider here the point-tangent arrangement of a basic surface segment and deduce those orientations which identify the surface equally. From this it is easily concluded that the entire surface $\Gamma$ is not altered if like orientation changes are made for each of its component pieces. For notational convenience, let $G_0$ be a basic surface segment defined by $G_0 = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{T}_a, \mathbf{T}_b, \mathbf{T}_c, \mathbf{T}_d, \mathbf{S}_a, \mathbf{S}_b, \mathbf{S}_c, \mathbf{S}_d)$. By comparing coefficients of like powers of the parameter variables in the equations for each of the following surfaces, we find that they form, along

TABLE 1

| | | | |
|---|---|---|---|
| 1 | $\mathbf{P}_{0,0} = (2, 5, 2)$ | 14 | $\mathbf{P}_{3,2} = (10.453, -1.473, 1.276)$ |
| 2 | $\mathbf{P}_{1,0} = (2.49, 2.549, 2.7)$ | 15 | $\mathbf{P}_{4,2} = (10.304, -2.965, 1.2)$ |
| 3 | $\mathbf{P}_{2,0} = (2.981, .097, 2.9)$ | 16 | $\mathbf{P}_{0,3} = (14.055, 3.503, 1)$ |
| 4 | $\mathbf{P}_{3,0} = (3.471, -2.354, 3)$ | 17 | $\mathbf{P}_{1,3} = (14.194, 2.261, .8)$ |
| 5 | $\mathbf{P}_{4,0} = (3.961, -4.806, 3)$ | 18 | $\mathbf{P}_{2,3} = (14.331, 1.018, 1.2)$ |
| 6 | $\mathbf{P}_{0,1} = (7.45, 4.001, 2)$ | 19 | $\mathbf{P}_{3,3} = (14.469, -.224, 1.6)$ |
| 7 | $\mathbf{P}_{1,1} = (7.35, 2.004, 3)$ | 20 | $\mathbf{P}_{4,3} = (14.607, -1.466, 1.25)$ |
| 8 | $\mathbf{P}_{2,1} = (7.251, .006, 2.5)$ | 21 | $\mathbf{P}_{0,4} = (15, 4, 0)$ |
| 9 | $\mathbf{P}_{3,1} = (7.151, -1.991, 1.5)$ | 22 | $\mathbf{P}_{1,4} = (15.720, 2.689, 0)$ |
| 10 | $\mathbf{P}_{4,1} = (7.051, -3.989, 2)$ | 23 | $\mathbf{P}_{2,4} = (16.457, 1.378, .2)$ |
| 11 | $\mathbf{P}_{0,2} = (10.901, 3.005, 1.5)$ | 24 | $\mathbf{P}_{3,4} = (17.185, .066, .6)$ |
| 12 | $\mathbf{P}_{1,2} = (10.751, 1.512, 1.425)$ | 25 | $\mathbf{P}_{4,4} = (17.914, -1.245, 1.2)$ |
| 13 | $\mathbf{P}_{2,2} = (10.602, .02, 1.35)$ | | |

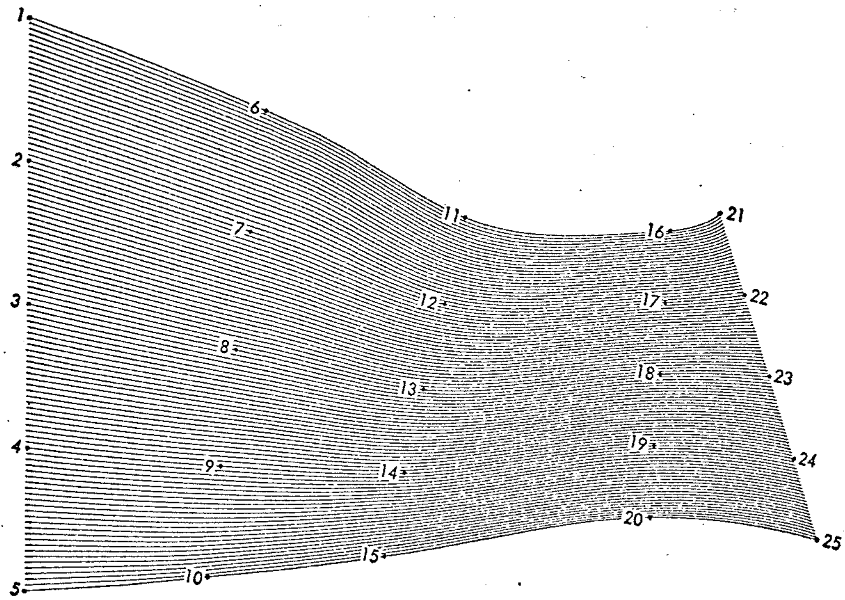



FIG. 5

with $G_0$, a complete system of eight identical surfaces (complete in the sense of exhaustion of all possible point-tangent orderings, allowing tangent reversals):

$$G_1 = (\mathbf{C}, \mathbf{D}, \mathbf{A}, \mathbf{B}, \mathbf{T}_c, \mathbf{T}_d, \mathbf{T}_a, \mathbf{T}_b, -\mathbf{S}_c, -\mathbf{S}_d, -\mathbf{S}_a, -\mathbf{S}_b)$$

$$G_2 = (\mathbf{B}, \mathbf{A}, \mathbf{D}, \mathbf{C}, -\mathbf{T}_b, -\mathbf{T}_a, -\mathbf{T}_d, -\mathbf{T}_c, \mathbf{S}_b, \mathbf{S}_a, \mathbf{S}_d, \mathbf{S}_c)$$

$$G_3 = (\mathbf{D}, \mathbf{C}, \mathbf{B}, \mathbf{A}, -\mathbf{T}_d, -\mathbf{T}_c, -\mathbf{T}_b, -\mathbf{T}_a, -\mathbf{S}_d, -\mathbf{S}_c, -\mathbf{S}_b, -\mathbf{S}_a)$$

$$G_4 = (\mathbf{C}, \mathbf{A}, \mathbf{D}, \mathbf{B}, -\mathbf{S}_c, -\mathbf{S}_a, -\mathbf{S}_d, -\mathbf{S}_b, \mathbf{T}_c, \mathbf{T}_a, \mathbf{T}_d, \mathbf{T}_b)$$

$$G_5 = (\mathbf{B}, \mathbf{D}, \mathbf{A}, \mathbf{C}, \mathbf{S}_b, \mathbf{S}_d, \mathbf{S}_a, \mathbf{S}_c, -\mathbf{T}_b, -\mathbf{T}_d, -\mathbf{T}_a, -\mathbf{T}_c)$$

$$G_6 = (\mathbf{D}, \mathbf{B}, \mathbf{C}, \mathbf{A}, \mathbf{S}_d, -\mathbf{S}_b, -\mathbf{S}_c, -\mathbf{S}_a, -\mathbf{T}_d, -\mathbf{T}_b, -\mathbf{T}_c, -\mathbf{T}_a)$$

$$G_7 = (\mathbf{A}, \mathbf{C}, \mathbf{B}, \mathbf{D}, \mathbf{S}_a, \mathbf{S}_c, \mathbf{S}_b, \mathbf{S}_d, \mathbf{T}_a, \mathbf{T}_c, \mathbf{T}_b, \mathbf{T}_d).$$


FIG. 6

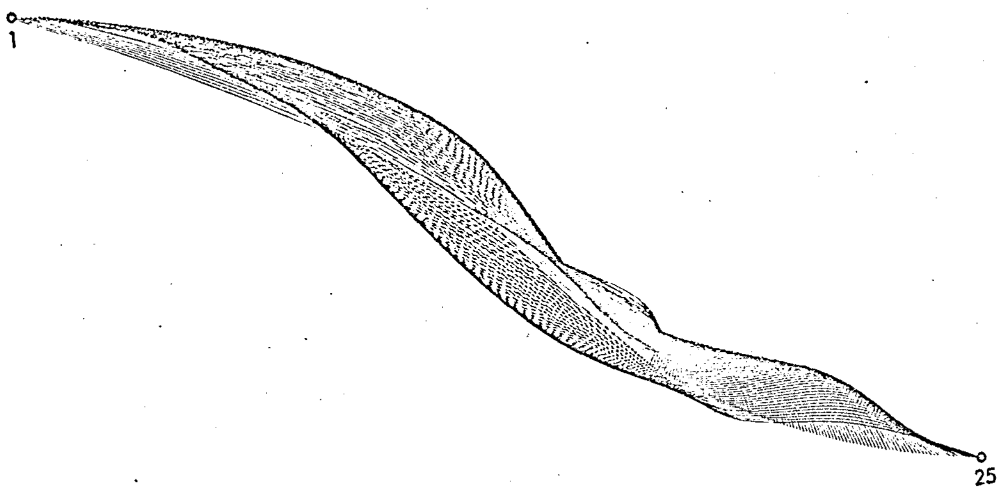
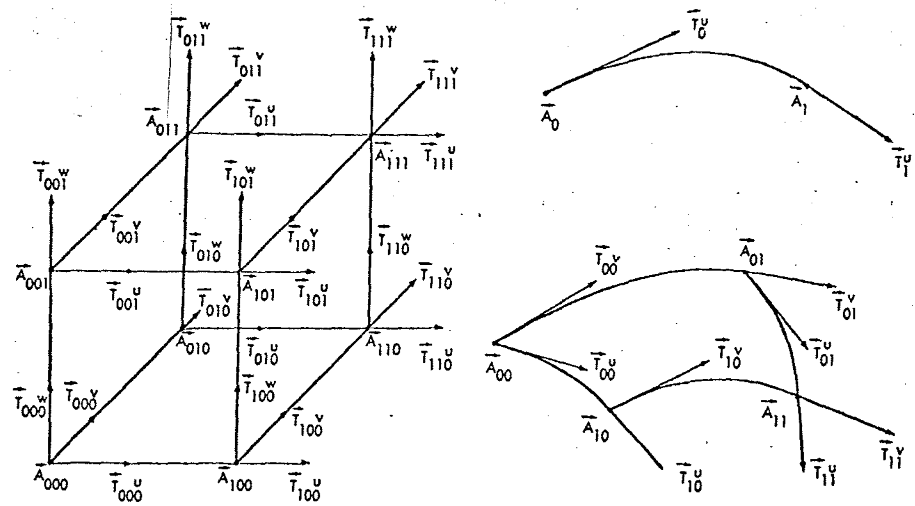



FIG. 7

*Example.* Twenty-five points $\mathbf{P}_{i,j}$, $i = 0, 1, 2, 3, 4$, $j = 0, 1, 2, 3, 4$, are listed in Table 1. Necessary computer programs were written using the methods of this paper to define a smooth surface through the points. The surface equations were used by these programs to generate a family of tabulated coordinate curves on the composite surface, which in turn were drawn by a numerically controlled drafting machine in the views of Figures 5, 6, and 7.

*Higher-Dimensional Generalizations*

The notions of the previous sections are preserved for this discussion. First, an $E^4$ (four-dimensional Euclidean space) vector-valued function is defined, whose domain is a unit cube in $E^3$, the corners of which map into a given 8-point array in $E^4$ such that the structure of lines joining adjacent points is equivalent to the cube-type structure of Figure 8. For convenience then, the discussion henceforth is notationally directed to this cube. At each $\mathbf{A}_{ijk}$ is defined (same as previous composite curve construction) three direction vectors $\mathbf{T}^u_{ijk}$, $\mathbf{T}^v_{ijk}$, $\mathbf{T}^w_{ijk}$ correspond-




FIGS. 8-10

ing to the edge directions of the $u, v, w$-cube at each corner. Keep in mind that we really desire an interpolation scheme in $E^4$, i.e. to find functions $\mathbf{P}(u, v, w)$ which smoothly map adjacent 8-pointed cubes into a rectangular-like parallelepiped net of points in $E^4$. The intention here is to generalize the results of the one- and two-dimensional solutions directly. To this end, first recall these equations. For notational convenience let $\zeta_u = [(2u + 1)(u - 1)^2, u^2(3 - 2u), u(u - 1)^2, u^2(u - 1)]$ and $\eta_u = [(2u + 1)(u - 1)^2, u^2(3 - 2u)]$. Then: (i) the curve equation can be written (Figure 9) as a dot product

$$\mathbf{P}(u) = (\mathbf{A}_0, \mathbf{A}_1, \mathbf{T}_0^u, \mathbf{T}_1^u)\cdot\zeta_u, \quad u \in [0, 1];$$

(ii) the two-dimensional surface segment equation is written as (Figure 10)

$$\mathbf{P}(u, v) = [\mathbf{P}_0(u), \mathbf{P}_1(u), (\mathbf{T}^v_{00}, \mathbf{T}^v_{10})\cdot\eta_u, (\mathbf{T}^v_{01}, \mathbf{T}^v_{11})\cdot\eta_u]\cdot\zeta_v, \quad u, v \in [0, 1],$$

where $\mathbf{P}_m(u) = (\mathbf{A}_{0m}, \mathbf{A}_{1m}, \mathbf{T}^u_{0m}, \mathbf{T}^u_{1m})\cdot\zeta_u, \quad (m = 0, 1)$.

By examining the forms of these equations, the following generalization is apparent (for $E^4$):

$$\begin{aligned}\mathbf{P}(u, v, w) = \{&\mathbf{P}_0(u, v), \mathbf{P}_1(u, v), [(\mathbf{T}^w_{000}, \mathbf{T}^w_{100})\cdot\eta_u, (\mathbf{T}^w_{010}, \mathbf{T}^w_{110})\cdot\eta_u]\cdot\eta_v,\\ &[(\mathbf{T}^w_{001}, \mathbf{T}^w_{101})\cdot\eta_u, (\mathbf{T}^w_{011}, \mathbf{T}^w_{111})\cdot\eta_u]\cdot\eta_u\}\cdot\zeta_w \quad u, v, w \in [0, 1],\end{aligned} \tag{10}$$

where $\mathbf{P}_0(u, v)$ and $\mathbf{P}_1(u, v)$ have interpretations similar to $\mathbf{P}_0(u)$ and $\mathbf{P}_1(u)$ above.

Now check that this equation satisfies the problem requirements. It can be shown by evaluating (10) at the cube corners that the initial conditions are satisfied:

(a) $\mathbf{P}(i, j, k) = \mathbf{A}_{ijk}$

(b) $Du\,\mathbf{P}(i, j, k) = \mathbf{T}^u_{ijk}, \quad (Du = \partial/\partial u)$,

(c) $Dv\,\mathbf{P}(i, j, k) = \mathbf{T}^v_{ijk}$

(d) $Dw\,\mathbf{P}(i, j, k) = \mathbf{T}^w_{ijk}, \quad i, j, k = 0, 1.$

The condition for smoothness is: at each point in a boundary face between two hypersurfaces of the composite surface, the three directions $Du\mathbf{P}$, $Dv\mathbf{P}$, $Dw\mathbf{P}$ are identically calculated. Thus, considering just one hypersurface segment $\mathbf{P}(u, v, w)$, the following relations must be shown:

$$(1)\ Dm\,\mathbf{P}(0, v, w) \overset{R}{=} Dm\,\mathbf{P}(1, v, w),$$

$$(2)\ Dm\,\mathbf{P}(u, 0, w) \overset{R}{=} Dm\,\mathbf{P}(u, 1, w),$$

$$(3)\ Dm\,\mathbf{P}(u, v, 0) \overset{R}{=} Dm\,\mathbf{P}(u, v, 1), \quad m = u, v, w,$$

where the symbol "R" restricts the equality to "after corresponding point-tangent substitutions are made between opposite faces." It is only a matter of arithmetic to show these to be true.

In a like manner, generalizations to $E^5$, $E^6$, and so forth, can be constructed and verified.

Applications for the higher-dimension surfaces can be found in regional point distributions where associated with each point is some physical intensity valuation, such as temperature, pressure, sound intensity, electric field strength, etc.

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