Geostatistical Ore Estimation — A Step-by-Step Case Study

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MICHEL DAVID,
Assistant Professor,
Department of Geological Engineering,
Ecole Polytechnique,
Montréal, Québec, Canada

ABSTRACT

The different concepts of geostatistics will be defined and each illustrated by a real application. Starting from the samples available, different variograms will be constructed and related to one another. The precision with which volume, tonnage and grades are known will be computed and charts giving the expected precision for a subsequent drilling grid will be derived. These charts will help in the decision of whether or not to proceed with the drilling program.

INTRODUCTION

THIS PAPER does not intend to demonstrate new results, in the theory of ore reserve estimation, but does wish to promote an already several-year-old method by showing all the information which can be obtained by rather elementary techniques, once a first campaign of drilling has been completed. We will first try to sum up what the principles are and then apply them one by one.

PRINCIPLE AND THEORY

The theoretical basis of Dr. Matheron's geostatistics has already been reviewed in a number of recently published papers[1],[2],[3],[4]. This theory, which is extensively used in Europe, South America and South Africa, considers that the grade of ore in a deposit behaves like a regionalized variable, which roughly means that it is neither a random variable — a sample near a rich one tends to be rich too — nor a continuous function — two very nearby samples can differ. We will recall briefly the other properties of regionalized variables.

First of all, they have a *support,* the grade at one point does not mean anything, and one needs to define a small volume (on which one takes the average of the grade) by its size, orientation and location. Next, the variables may be more or less *continuous;* iron ore will be rather continuous, for instance, whereas gold will be completely discontinuous, presenting what we call a nugget effect. Thirdly, this continuity may not be the same in all directions; the zone of influence of a sample does not have the same extent in all directions, and the variable presents *anisotropies.* Lastly, the variables usually present a *structure:* the dependence of samples on one another decreases with their distance and sometimes reaches independence after a finite range.

All these four properties are essential for any precision calculation of average grade and tonnage of ore.

How to Take Them Into Account

These four characteristics are easily visualized and quantified by a single instrument — the variogram.

The variogram is a function expressing the variance of two samples with respect to their distance and is represented by the following formula:

$$2\,\gamma(h) = \frac{1}{D}\int_D [f(x) - f(x+h)]^2 dx$$

where D is the field on which the variable f(x) is defined and h is a variable, which is the distance between two points.

We will not dwell any longer on the properties of this now well-known function and will refer our reader to previous publications. Thus, the first thing to do is to construct the variogram of the variable one wishes to study. What is next?

The Estimation Problem Itself

Given a block of ore, one has the choice of assigning it the grade of its sample — what would the error then be? — or assigning it a weighted average of available samples inside and outside the block. Then, are there any weights which are better than others? Can we compute them?

Dr. G. Matheron has already given the answer to all these problems, and we will now show which are the basic simple formulae which will enable us to reach our goal. Once again, we will not demonstrate any formula, but refer specialists who want to study the question in depth to Dr. Matheron's thesis[6].

A CASE STUDY

The deposit we are going to study is a copper-nickel sulphide deposit. Roughly speaking, a pegmatite fills a syncline and thus the orebody is more or less an elongated ellipsoid. *Figures 1* and *2* show longitudinal and vertical sections.

The samples available and working variables: After the discovery hole was bored, a regular pattern of vertical drill-holes was laid out. The surface plan of the property is shown in *Figure 3.* The problem we want to study is a three-dimensional one, but we feel that this map gives us sufficient information for a global estimation. In other words, we can easily reduce our problem to a two-dimensional one by using a variable which we call "accumulation" and which is the product, for each vertical drill-hole, of the mineralized length and the grade. This variable is proportional to the variable we are interested in. We will not sell grade, nor tonnage, but a combination of both: a quantity of metal. Thus, our working variables will be the accumulation and length of intersections, the grade being considered as a secondary variable obtained as the ratio of the first two.

The Questions We Want to Answer

Usually, for a deposit of this kind, we want to know the tonnage and the average grade, and the associated precision. These are the final figures we want to arrive at. In fact, there are prerequisites, and these are described below.

The Geometric Problem

In our case, we have ore and sediments. Thus, the distinction between ore and waste is clear and there is an answer to the question "what is the ore volume?" Given a tonnage factor, which is itself a regionalized variable, we will work out the tonnage of ore. This will be called the geometric problem.

Quantity of Metal: Dollar Value of the Deposit

The estimation of the quantity of metal itself will not be a problem, because we have a reglar grid of samples. Working out a mean value will be simple arithmetic.

Precision Computations

The computation of precision will require far more attention, and this will be the main contribution of geostatistics. Isn't it in fact the most important question for the company as well?

To sum up, we will have three basic regionalized variables to work with:

s = surface of the mineralized area in horizontal projection;

h = thickness of the mineralized area vertically measured;

a = accumulation of metal, which is a variable in a horizonta plan and expresses the grade times the thickness of ore.

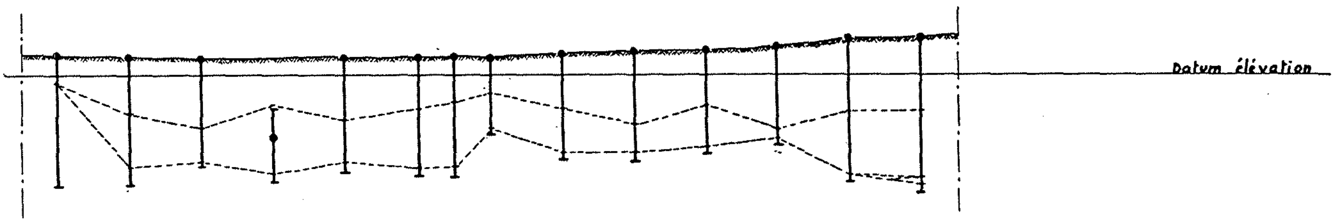
Thus, we will first study them by their variograms, and obtain their mean value and precision. Next, we will consider that:

V = the volume of ore = $s \times \bar{h}$

Q = the metal quantity = $\bar{a} \times \bar{h} \times$ constant

$ = the dollar value of ore

t = the grade = a/h



section 2+00S

FIGURE 1 — Vertical section — Expo-Ungava Property.

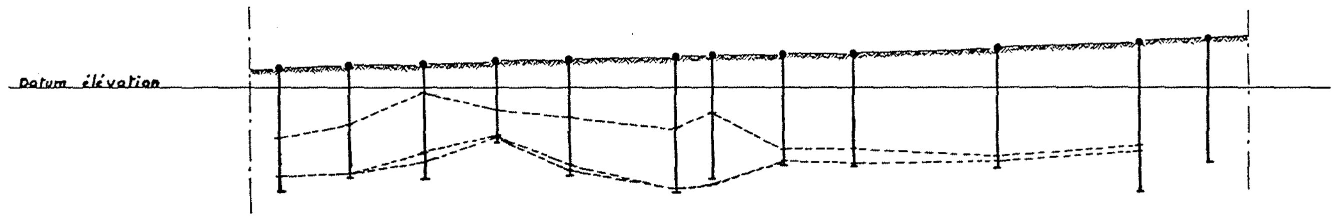


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FIGURE 2 — Vertical section — Expo-Ungava Property.

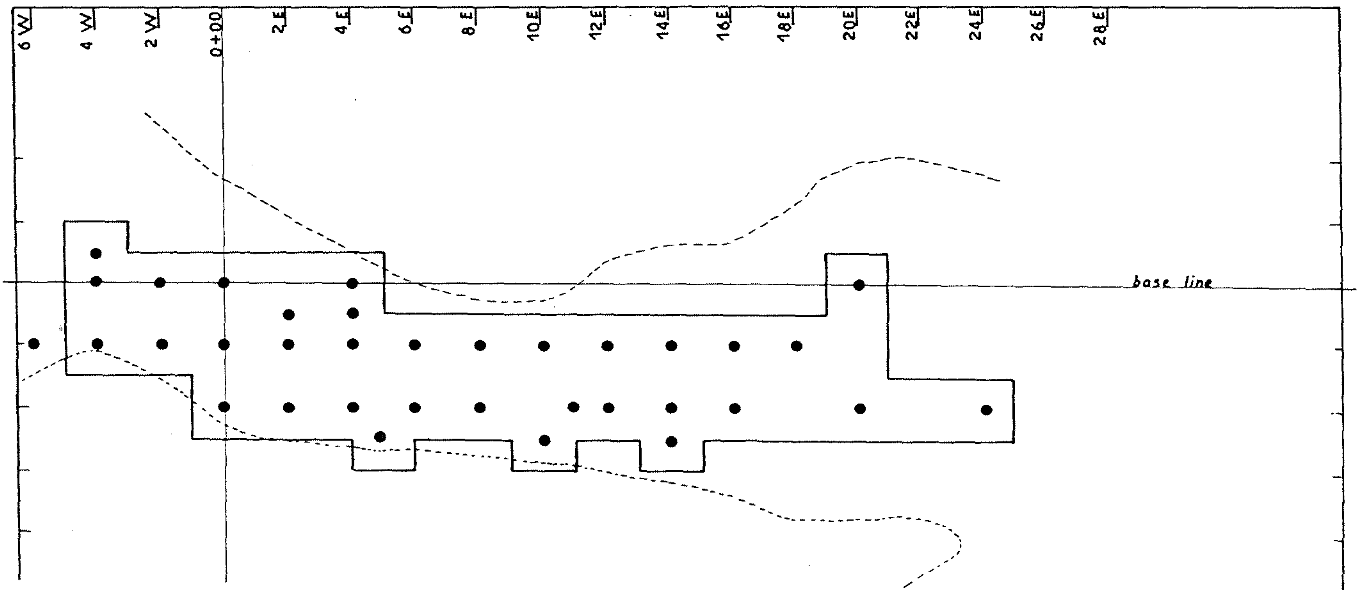


FIGURE 3 — Plan showing drill pattern at the Expo-Ungava Property.

All these computations will be done for the actual drilling grid, but what may be still more interesting is that *we will be able to see what another grid would give for precision.*

The Variogram

Our first two basic variables will be the accumulation of copper and of nickel. The two variograms have been constructed in the longitudinal direction. They have been plotted on *Figure 4,* and both of them appear to be de Wijsian. This means that they can be described by an equation of the form:

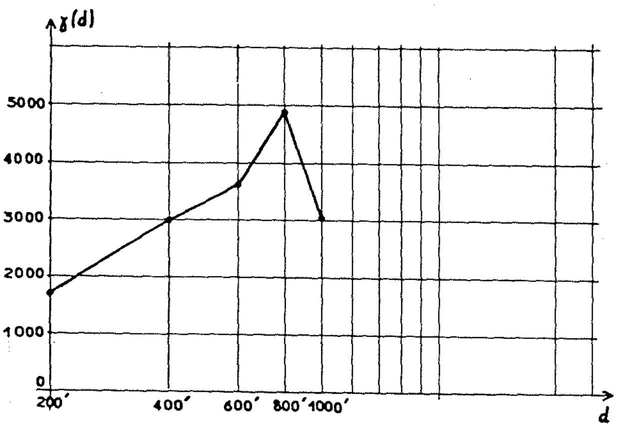
$$\gamma(d) = 3\alpha\left(\mathrm{Log}\,\frac{d}{l} + \frac{3}{2}\right)$$

where l is the average length of the holes and d the distance between holes; 3α is the coefficient of intrinsic dispersion and characterizes the variability of the mineralization.

This equation may seem strange at first glance. Why this coefficient 3/2? In fact, this equation is a form of the equation of the variogram of holes of average length l when the underlying punctual variogram has the simple form:

$$\gamma(d) = 3\,\alpha\,\mathrm{Log}\,d$$

Let us take this opportunity to recall that given the variogram of one kind of sample, it is possible to obtain the variogram of any other sample by an algorithm which Matheron calls "montée" and which we can consider as a regularization. This very important problem is solved in reference (6), p. 28.



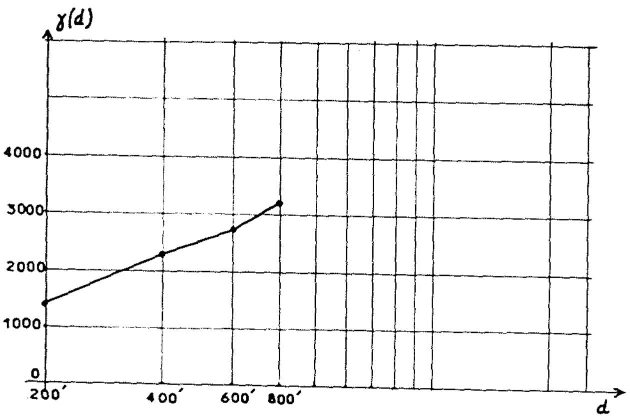


FIGURE 4 — Copper and nickel variograms for the Expo-Ungava Property.

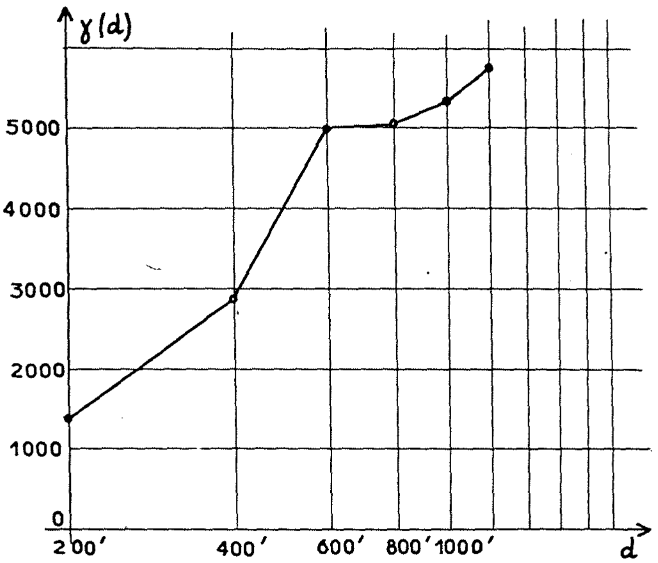


FIGURE 5 — Thickness variogram (W-E) at Expo-Ungava.

Reaching this point, we will be able to determine whether or not there is anisotropy. This means that we will be able to say how deep one has to go to obtain the same variation vertically as horizontally, or how many times quicker the grade changes in the vertical direction as compared to the horizontal one. Let us now see the numerical results.

The variogram of *copper* accumulation can be fitted to the equation:

$$\gamma(d) = 1900 + 1820\,\log_e d$$

where the grid size (200 ft) is taken as a unit of distance. We can rewrite this:

$$\gamma(d) = 1820\left(\log_e \frac{d}{1.58} + \frac{3}{2}\right)$$

which gives $3\,\alpha = 1820$ and reveals that the average length of the holes is equivalent to 1.58 times the grid unit:

$$h = 1.58 \times 200' = 316'$$

As we know that the true average length is 136.6 ft, we deduce that the vertical anisotropy coefficient is

$$\frac{316}{136.6} = 2.31$$

The zone of influence of a sample is more than two times wider than its depth. This in fact quantifies a notion which is familiar to geologists.

For *nickel,* we obtain a variogram of the same kind, which gives:

$$3\alpha = 1300$$

and yields a similar anisotropy coefficient. Had we found different ratios for copper and nickel, it would have raised serious genetic problems!

We also need the variogram of thicknesses, as these values are certainly not independent of each other. Here again we have a De Wijsian variogram, the slope of which is $3\,\alpha = 2360$ (*Figure 5*).

We now reach our last basic variable, the surface. The geometric variogram has such properties that we don't even need to compute it; we can directly reach the precision of our estimation of the surface by means of simple parameters.

ESTIMATION PROBLEMS

The Geometric Problem

To obtain an estimate of the surface, we simply assign to each mineralized hole a square of influence and we count 34 such squares, so that our estimate of the area — which might seem crude — but still bears no meaning without a variance, is:

$$S = 34 \times 200 \times 200 = 1{,}360{,}000 \text{ square feet.}$$

The problem is now to put a confidence range on that area. Is it 1,360,000 ± 5 per cent or ± 20 per cent? The answer is given by the following formula by Matheron[7],[10], which is deduced from the transitive theory [6,108].

$$\frac{\sigma_s^2}{s^2} = \frac{1}{n^2}\left[\frac{N_1}{6} + 0.0609\frac{N_2^2}{N_1}\right] \ldots\ldots\ldots\ldots N_2 \geqslant N_1$$

where:

$\frac{\sigma_s^2}{s^2}$ is the relative variance of the surface

$a_1 \times a_2$ is the grid surface

s is the estimated surface

n is such that $s = n \times a_1 \times a_2$

N_1 is the total length of border, parallel to the north-south direction, divided by $2a_1$.

N_2 is the same for the east-west direction, divided by $2a_2$.

This formula will give us the opportunity to point out another essential principle of geostatistics, which states that a global estimation variance can be obtained by the combination of a "line term" and a "section term". More precisely[6,87], the estimation of maximum density lines, from samples, can be considered as independent of the estimation of a section (plan) by these lines. The two successive extensions (grade of a point into a line, and grade of the line into a rectangle) result in independent variances.

This yields, obtaining N_1 and N_2 from *Figure 3*:

$$\frac{\sigma_s^2}{s^2} = 0.0028 \text{ or } \frac{\sigma_s}{s} = 0.05$$

which in clearer (?) terms means that we have 95 per cent chances that the surface we are estimating is 1,360,000 ± 10 per cent approximately.

Estimation of Thickness

When one has a deposit regularly drilled one has an estimation variance (where R is such that $\pi R^2 = a^2$):

$$\sigma_E^2 = 3\alpha\left[1.344\,\frac{R}{h} - \frac{5\,R^2}{12\,h^2} + \frac{1}{12}\,\frac{R^4}{h^4}\ldots\right]$$

(valid for $h > 2R$)

which is tabulated in Matheron[7,304] and gives:

$$\frac{\sigma_h^2}{h^2} = 0.0042$$

Average Accumulation Estimation

Let V be the volume of the deposit and H the total length drilled. Matheron has shown that the estimation variance of the accumulation is

$$\sigma_E^2 = 3\alpha.\,0.76\sqrt{\frac{V}{H^3}} \ldots\ldots \text{Matheron (7, 306)}$$

Once again, this formula is obtained from the "variance composition" principle.

In our case, taking into account our anisotropy ratio, we obtain:

for nickel.......... $\sigma_e^2 = 18.50$

for copper.......... $\sigma_e = 25.90$

To this we must add the incidence of the geometric definition of the surface, which is not perfect. This adds a term, which is:

$$\sigma_{\text{sup}}^2 = \sigma_a^2 \cdot \frac{\sigma_s^2}{s^2} \ldots\ldots\ldots\ldots\ldots\ldots \text{Formery (5, 179)}$$

where σ_s^2/s^2 has already been computed and where σ_a^2 is the variance of the variable we are working on.

Thus, reaching the relative variances we have:

$$\frac{D^2(a)}{a^2} = \frac{\sigma_{ea}^2}{a^2} + \frac{\sigma_a^2}{a^2}\cdot\frac{\sigma_s^2}{s^2}$$

which gives a relative precision of 0.073 for nickel and 0.086 for copper.

This concludes all the preliminary computations and what follows is only trivial arithmetic. We obtain all the precisions we want to know, thanks to our three basic variables.

The volume (V) is $V = h \times s$, and thus one has approximately:

$$\frac{\sigma^2(v)}{v^2} = \frac{D^2(h)}{h^2} + \frac{\sigma_s^2}{s^2}$$

or a relative precision of $\frac{\sigma(v)}{v} = 0.08$

The metal quantity is proportional to the product a x s, thus

$$\frac{\sigma^2(Q)}{Q^2} \backsimeq \frac{D^2(a)}{a^2} + \frac{\sigma_s^2}{s^2}$$

for nickel........... $\frac{\sigma(Q)}{Q} = 0.090$

for copper.......... $\frac{\sigma(Q)}{Q} = 0.096$

Note that this, in fact, together with the correlation coefficient between copper and nickel, will provide us with what we are interested in; that is, the precision on the dollar value of the deposit.

We obtain the precision on the grade by

$$\bar{t} = \bar{a}/\bar{h}$$

where $\bar{t}$ is the average grade we are looking for

$\bar{a}$ is the average accumulation

$\bar{h}$ is the average thickness.

We thus have:

$$\frac{\sigma^2(\bar{t})}{\bar{t}^2} \backsimeq \frac{\sigma_a^2}{\bar{a}^2} + \frac{\sigma_h^2}{\bar{h}^2} - 2\,\rho_{ah}\,\frac{\sigma_h}{h}\,\frac{\sigma_a}{a}$$

and we obtain in this way a relative precision

$$\frac{\sigma t}{t} = 0.075 \text{ for copper}$$

$$\frac{\sigma_t}{t} = 0.070 \text{ for nickel}$$

What Would Another Grid Have Given?

An obvious question at this stage is: what would have been the effect of increasing or decreasing the number of holes drilled? To make this apparent, we will express all the variances we have computed in the previous paragraph

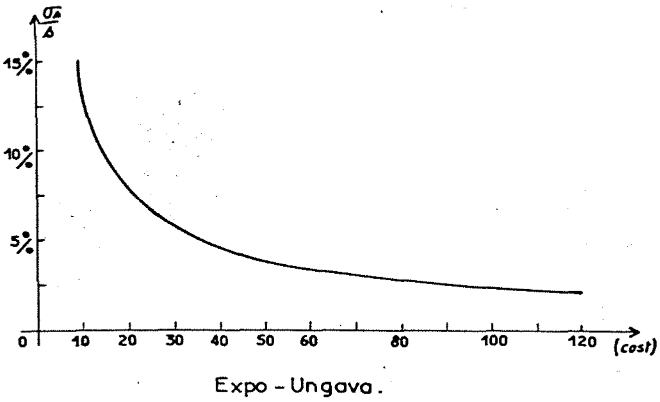


FIGURE 6 — Precision as a function of cost for SURFACE estimation (Expo-Ungava).

as functions of n (the number of holes drilled), and we will obtain a set of values which will allow us to select the grid size according to the precision we want. We will quickly review all these formulae.

Surface Variance

The general formula for a square grid is, in this case:

$$\frac{\sigma_s^2}{s^2} = 0.55 \cdot \frac{1}{n^{3/2}}$$

which gives the curves of *Figure 6.*

In our particular case, it seems *a priori* obvious that a rectangular grid is better than a square one. Let us see how much better a grid of 200 by 400 ft would have been —

$$\frac{\sigma_s^2}{s^2} = 0.0054, \text{ or a precision of } \pm 15 \text{ per cent;}$$

a grid of 200 by 600 ft —

$$\frac{\sigma_s^2}{s^2} = 0.0105, \text{ or a precision of } \pm 20 \text{ per cent;}$$

a grid of 300 by 600 ft —

$$\frac{\sigma_s^2}{s^2} = 0.016, \text{ or a precision of } \pm 25 \text{ per cent.}$$

This can help us to decide which grid to use for further drilling, in the same type of serpentine, for volume definition.

Thickness Variance

As thickness can be regarded as an accumulation in an isotropic deposit of grade one, we can apply Matheron's formula which gives:

$$\sigma_h^2 = 3\alpha \cdot \frac{0.76}{\bar{h}} \sqrt{\frac{S}{n^3}}$$

or, with our values:

$$\frac{\sigma_h^2}{h^2} = \frac{0.83}{n^{3/2}}$$

Accumulation Variance

Following the same formula, we find the average value of accumulation —

for copper: $\frac{\sigma_a^2}{a^2} = \frac{0.95}{n^{3/2}}$

for nickel: $\frac{\sigma_a^2}{a^2} = \frac{0.735}{n^{3/2}}$

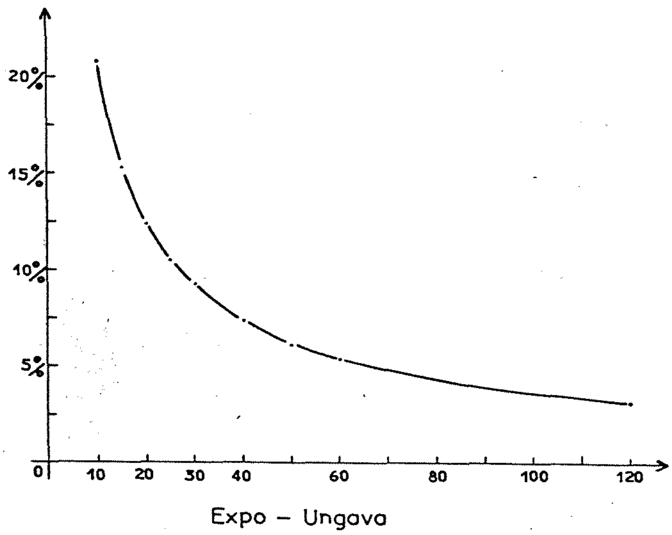


FIGURE 7 — Precision as a function of cost for VOLUME estimation (Expo-Ungava).

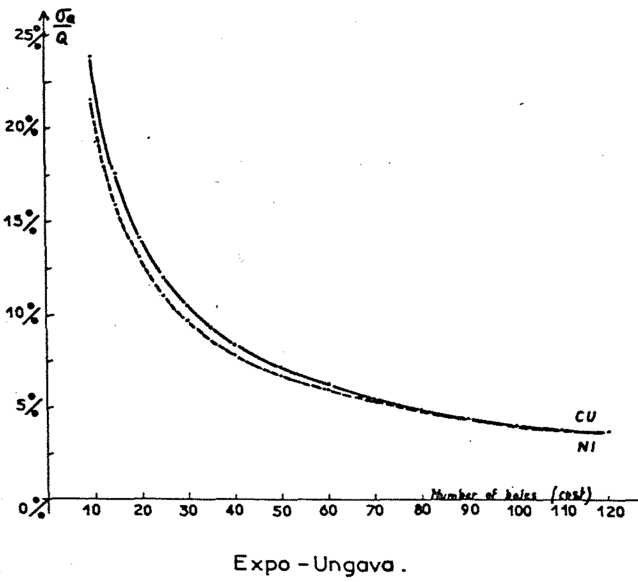


FIGURE 8 — Precision as a function of cost for NICKEL and COPPER quantity.

We have to add to these terms the effect of the uncertainty on the surface. These terms are —

for nickel: $.475 \frac{\sigma_s^2}{s}$

for copper: $.545 \frac{\sigma_s^2}{s^2}$

or finally, replacing $\frac{\sigma_s^2}{s^2}$ by its value, we obtain —

for copper: $\frac{\sigma_a^2}{a^2} = \frac{1.25}{n^{3/2}}$

for nickel: $\frac{\sigma_a^2}{a^2} = \frac{0.997}{n^{3/2}}$

Variance of the Volume

We repeat again the reasoning and obtain:

$$\frac{\sigma^2(v)}{V^2} = \frac{.83}{n^{3/2}} + \frac{.55}{n^{3/2}} = \frac{1.38}{n^{3/2}},$$

which gives the curve of *Figure 7* for relative precision.

Variance of the Metal Quantity

We have again —

$$\text{for copper: } \frac{\sigma_Q^2}{Q^2} = \frac{1.25}{n^{3/2}} + \frac{.55}{n^{3/2}} = \frac{1.8}{n^{3/2}}$$

$$\text{for nickel: } \frac{\sigma_Q^2}{Q^2} = \frac{1}{n^{3/2}} + \frac{.55}{n^{3/2}} = \frac{1.55}{n^{3/2}}$$

which can be represented on the curves of *Figure 8*.

As a conclusion to these variance calculations, we will note that all of them can be expressed by a formula of the type:

$$\sigma^2 = An^{-3/2}$$

whereas the application of standard statistics would have suggested a variance of the type:

$$\sigma^2 = \frac{A}{n}$$

BLOCK-GRADING, KRIGING

The problem of optimum weighting factors for block-grading has been completely solved by Matheron by means of the kriging technique[6,214], and we can thus obtain the best estimate for the grade of a block, given the grade of some nearby intersections.

Optimal Weighting Factors for Large Blocks

We will consider the following problem.

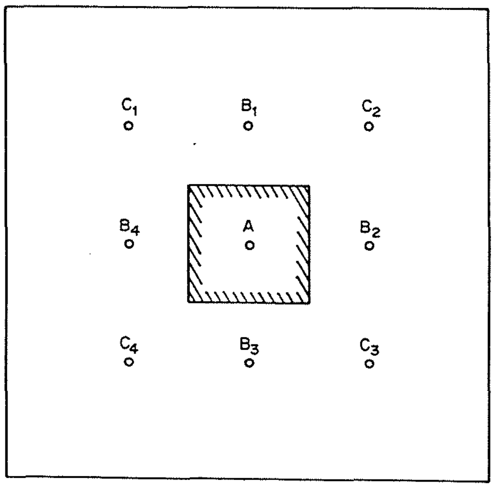


FIGURE 9 — Block-grading.

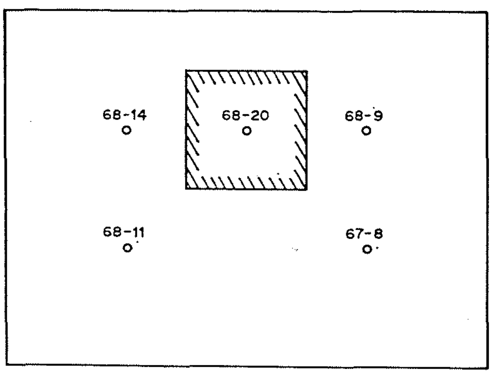


FIGURE 10 — Example for block-grading.

Given a block of 200 by 200 ft, and the grade of the nearest grid points, how would we weight them to grade the block? The grade of the central block will be obtained by a linear estimate of the form:

$$Z = (1 - \lambda - \mu)\, u + \lambda v + \mu w$$

where u is the grade of the central hole

v is the average grade of the first "round"

w is the average grade of the second "round"

and $(1 - \lambda - \mu)$, λ, μ are their respective weights *(Figure 9)*.

These weights are uniquely determined by the two following conditions. They add up the one in order to obtain the same average for blocks and for samples. They should minimize the estimation variance (a function of the variogram).

$$D^2\ (Z - Z^*) \text{ minimum.}$$

This condition yields a system of linear equations in which the unknowns are the weights we are looking for.

To avoid solving this system for each block, a set of charts has been produced by which one reads the weights and related precision (kriging variance), given the configuration of samples.

Let us take as an example the following particular case (*Figure 10*). This configuration is shown on *Figure 11*.

We have to read, on the horizontal axis, a ratio which is the average thickness over the grid side. Here it is 136/200, but we have seen that, due to anisotropy, 136 ft is equivalent to 316 ft, or that the ratio is thus 1.58. We thus read on the chart:

$$\lambda = 0.27$$
$$\mu = 0.13$$
$$1 - \lambda - \mu = 0.60$$

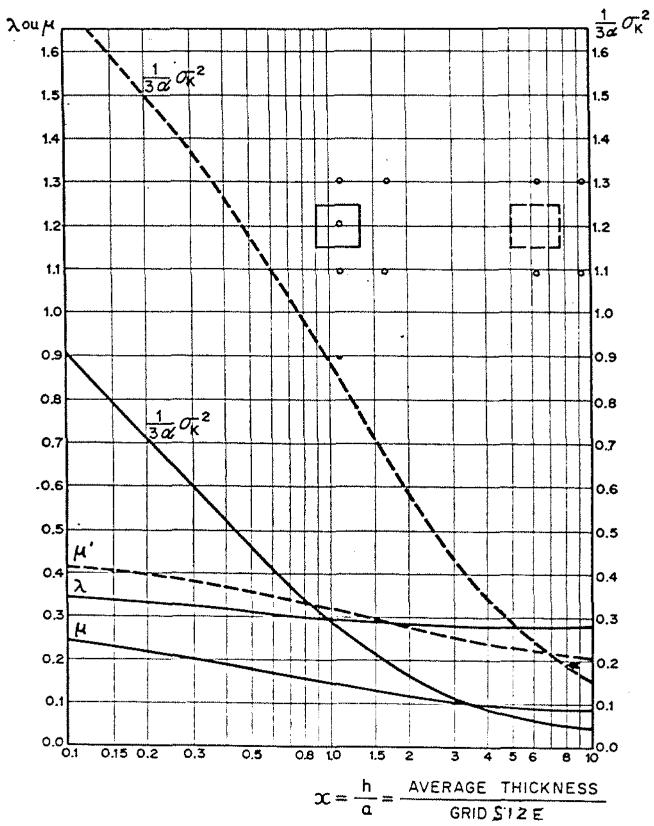


FIGURE 11 — Kriging chart.

Because the accumulations are:

for 68-14. . . . . 102.26
68-9. . . . . 44.64
68-11. . . . . 79.24
67-8. . . . . .149.0
68-20. . . . . 21.92

we have:

$$Z = (0.60)\ (21.9) + (0.27)\ \frac{(44.6 + 102.2)}{2} + (0.13)\ \frac{(79.2 + 149.0)}{2} = 47.79$$

instead of 21.9.

When giving this block the aforementioned Z grade, the variance is given by the same chart, on which we read:

$$\frac{1}{3\alpha}\,\sigma^2{}_k = 0.22$$

which makes: = 0.22 x 1800 = 260;

compared tc 18.50 x 34 if we had not weighted our values, or 629.00.

CONCLUSION

We have seen the sum of the *quantified* information which can be obtained — at a very low cost — from a regular sampling grid when one first is able to formulate the problems one wants to solve and secondly is able to use precise concepts and tools. The theory developed by Matheron and his team in Fontainebleau does provide these concepts for simple cases such as the one we have seen here, or for more intricate cases, where a clear understanding of the problems is the most important thing when coping with a large quantity of data.

How many times has a large quantity of samples and measurements been handled, tossed and shuffled by a computer without any result, for the simple reason that no firmly established theory was fed in at the same time?

ACKNOWLEDGMENT

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Bernard Nicolet, of AMAX Exploration, who was manager of the Expo Ungava project, initiated this research, and his help, comments and querries were a constant encouragement at the time of this study.

APPENDIX

Glossary of Geostatistical Terms

Here is a short list of terms which have a very precise sense in geostatistics and which one should make sure to fully understand before attempting any application to Dr. Matheron's theory.

These terms have been approximately defined in the previous section. Their precise definition can be found in Dr. Matheron's thesis, or on page 375 sq. of Dr. Serra's thesis.

— accumulation
— anisotropy
— estimation variance
— geometric problem
— kriging
— "montée"
— nugget effect
— range
— regionalized variable
— regularization
— support
— variogram

REFERENCES

(1) Blais, R., and Carlier, P., (1968), "Application of Geostatistics in Ore Valuation"; *CIM Special Volume No. 9*, 1968.
(2) Borgman, L., (1969), "Betting on Your Guesses"; 2e conférence de prise de décision dans l'industrie minérale, Vancouver, 1969.
(3) Bubenicek, L., and Haas, A., (1969), "Method of Calculation of the Iron Ore Reserves in the Lorraine Deposit"; *A Decade of Digital Computing in the Mineral Industry*, Alfred Weiss, editor, pp. 179-210.
(4) David, M., (1969), "The Notion of Extension Variance and its Application to the Grade Estimation of Stratiform Deposit"; Salt Lake City Symposium, 1969, AIME, Special Volume: *A Decade of Digital Computing.*
(5) Formery, P., (1966), "Unpublished Course of Geostatistics"; École des Mines, Nancy.
(6) Matheron, G., (1965), *Les variables régionalisées et leur estimation*; 308 pages, Masson, Paris.
(7) Matheron, G., (1969), "Traité de géostatistique"; Moscow.
(8) Matheron, G., (1963), "Principles of Geostatistics"; *Economic Geology*, Vol. 58, pp. 1246-1266.
(9) Matheron, G., (1967), "Kriging, or Polynomial Interpolation Procedures"; *CIM Bulletin*, Vol. 60, No. 665, pp. 1041-1045.
(10) Matheron, G., (1969), "Cours de géostatistique"; Cahier du Centre de Morphologie Mathématique de Fontainebleau, No. 1.
(11) Matheron, G., (1969), "Le Krigeage universel", Cahier du Centre de Morphologie Mathématique de Fontainebleau, No. 2.
(12) Matheron, G., (1969), "Théorie des ensembles aléatoires", Cahier du Centre de Morphologie Mathématique de Fontainebleau, No. 4.
(13) Serra, J., (1966), "Règles d'échantillonnage des minerais Lorrains en place"; Bulletin technique des mines de fer, Briey 84, 168-182.
(14) Serra, J., (1967), "Echantillonnage et estimation locale des phénomènes de transition miniers"; 690 p., Doctor Engineer thesis, Nancy.
(15) Serra, J., (1968), "Un critère nouveau de découverte des structures: le variogramme", in *Sciences des la Terre*, Nancy.
(16) Serra, J., (1968), "Morphologie mathématique et genèse des concrétions carbonates du Toarcien de Lorraine", in *Sedimentology*.
(17) Serra, J., (1968), "Les structures gigognes, morphologie mathématique et interprétation métallogénique", *Mineral. Deposita* (Berlin), 3-135-154.