VOL. 2 - NOS. 3 & 4 GEOTHERMICS SEPTEMBER DECEMBER 1973

International Journal of Geothermal Research

Galerkin Finite-Element Simulation of a Geothermal Reservoir

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ABSTRACT

The equations describing fluid flow and energy transport in a porous medium can be used to formulate a mathematical model capable of simulating the transient response of a hot-water geothermal reservoir. The resulting equations can be solved accurately and efficiently using a numerical scheme which combines the finite element approach with the Galerkin method of approximation. Application of this numerical model to the Wairakei geothermal field demonstrates that hot-water geothermal fields can be simulated using numerical techniques currently available and under development.

Introduction

While considerable effort is being expended in developing methodology for locating and developing geothermal reservoirs, relatively little attention is being directed toward the problem of evaluating long term reservoir performance. Because of the obvious role accurate reservoir performance predictions play in economic and environmental evaluation, we initiated two years ago a long term program to develop an accurate and flexible geothermal reservoir simulator. This paper is a brief report on the progress we have made in achieving this goal.

Drawing on earlier experience in the numerical simulation of groundwater flow and contaminant transport, we recognized that the geothermal reservoir is an exceedingly complex system and would be very difficult to simulate accurately using existing numerical methods. Moreover we realized that a clear understanding of the geology as well as the hydrology of this system was essential to the formulation of an accurate simulation model. Accordingly we found it necessary not only to develop new mathematical methodology but also to further our understanding of the geothermal reservoir. To achieve this dual objective we elected to develop a single phase hot-water model first and to apply it to a geothermal system where an accurate historical record of reservoir response to development could be used to demonstrate the ability of our model to simulate a field situation. Because the geothermal field at Wairakei, New Zealand, fulfilled our requirements for a prototype system during its first ten years of development, arrangements were made in co-operation with the New Zealand government to obtain performance records on this reservoir. In the following discussion we will outline the theoretical foundation of our model and demonstrate its ability to simulate the Wairakei system.

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Theoretical development

To simulate the response of a geothermal reservoir to exploitation, it is necessary to solve the equations describing fluid flow and energy transport in porous media. Whereas the flow equation is relatively easy to solve numerically, an accurate solution to the energy transport equation is a challenging mathematical problem. It has been well documented (PRICE ET AL. 1968) that finite difference methods, which have found widespread application in petroleum engineering and groundwater hydrology, lead to unacceptable numerical dispersion when applied to the transport equations. To circumvent this difficulty the method of characteristics was used to simulate the movement of contaminants in the subsurface (PINDER, COOPER 1970; BREDEHOEFT, PINDER 1973). Although satisfactory solutions were obtained for the problems considered, many important field problems are not amenable to this form of analysis and a more general approach based on a Galerkin-finite element scheme was developed (PINDER, FRIND 1972; PINDER 1973).

EQUATIONS

In generating a set of equations capable of describing energy transport in a system as complex as a geothermal reservoir, it is necessary to combine fundamental conservation laws, constitutive relationships, and equations of state. In general one must consider laws governing the *conservation of mass, momentum, and energy* which may be written using vector notation as

$$\frac{\partial \rho^*}{\partial t} + \nabla \cdot \rho^* \bar{v} - r = 0 \qquad (1)$$

$$\rho^* \frac{D\bar{v}}{Dt} - \nabla \cdot \bar{\bar{\pi}} - \rho^* \bar{F} - \bar{m} + r(\bar{v} - \bar{v}') = 0 \qquad (2)$$

and

$$\rho^* \frac{DU}{Dt} + \nabla \cdot \bar{h} - \pi : \nabla \bar{v} - \rho^* Q - \frac{r \theta p'}{\rho^{*'}} + r(U - U)$$

$$+ \frac{r}{2}(v^2 - v') - r\bar{v} \cdot (\bar{v} - \bar{v}') = 0 \qquad (3)$$

where D/Dt is the substantial time derivative and the variables are defined in the list of symbols. Several *constitutive* relationships must be introduced to aid in describing the dynamics of the geothermal reservoir system. These relationships include *Darcy's law*, which may also be deduced from equation (2)

$$\bar{v}_L = -\frac{\bar{\bar{k}}}{\mu \theta} \cdot (\nabla p - \rho g \nabla d), \text{ and}$$

a modified form of *Fourier's law*,

$$\bar{h}_L = -\theta \bar{K}^*_L \cdot \nabla T$$

where

$$\bar{K}_L^* = \bar{K}_L + K_d \bar{\bar{\psi}}$$

The principal *equations of state* used in this model relate viscosity to fluid temperature

$$\frac{1}{\mu} = 5380 + 3800A - 260A^3$$

where

$$A = (T_L - 150)/100$$

and fluid density to temperature and pressure

$$\rho_L = \rho_{0L} + \left(\frac{\partial \rho_L}{\partial T_{L_0}}\right)(T_L - T_0) + \left(\frac{\partial \rho_L}{\partial p_{L_0}}\right)(P_L - P_0)$$

Combining the conservation laws, constitutive relationships, and equations of state, and making generally accepted assumptions regarding second order terms in the equations, we obtain the fluid flow and heat transport equations for a single-phase geothermal system. In solving equations of this kind for petroleum and groundwater reservoirs, economic considerations have generally dictated the use of a two-dimensional model. In this first stage of our research effort we consider an areal two-dimensional model of the Wairakei system. An areal rather than vertical model was selected because we were principally concerned with predicting reservoir performance rather than examining the interesting scientific questions relating to cellular convection. The two-dimensional form of the fluid flow equation used in this simulator is

$$L_F(p, T) = \frac{\partial}{\partial x}\left[\frac{b\rho_L k_{xx}}{\mu_L}\left(\frac{\partial p}{\partial x} - \rho_L g \frac{\partial d}{\partial x}\right)\right] + \frac{\partial}{\partial y}$$

$$\left[\frac{b\rho_L k_{yy}}{\mu_L}\left(\frac{\partial p}{\partial y} - \rho_L g \frac{\partial d}{\partial y}\right)\right]$$

$$- b(\rho_L \alpha + \theta \rho_0 \beta_p)\frac{\partial p}{\partial t} - b\theta \rho_0 \beta_T \frac{\partial T}{\partial t} + br_L = 0 \qquad (4)$$

where the principal components of the permeability tensor are assumed colinear with co-ordinate axes. The corresponding energy transport equation is

$$L_T(p, T) = b[\theta \rho_L C_{vL} + (1-\theta)\rho_S C_{vS}]\frac{\partial T}{\partial t} + b\theta\rho_L$$

$$C_{vL}\left(v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y}\right) - \frac{\partial}{\partial x} b[\theta K^*_{xxL} + (1-\theta)K_{xxS}]$$

$$\frac{\partial T}{\partial x} - \frac{\partial}{\partial y} b[\theta K^*_{yyL} + (1-\theta)K_{yyS}]\frac{\partial T}{\partial y} - \frac{\partial}{\partial x} b[\theta K^*_{xyL}$$

$$+ (1-\theta)K_{xyS}]\frac{\partial T}{\partial y} - \frac{\partial}{\partial y} b[\theta K^*_{yxL} + (1-\theta)K_{yxS}]$$

$$\frac{\partial T}{\partial x} - b\theta\rho_L Q_L + b(1-\theta)\rho_S Q_S + b\left[\theta\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)\right.$$

$$\left. - \frac{r_i}{\rho'_L}\right]\frac{\beta}{k} T + r_L C_{vL}(T - T') = 0 \qquad (5)$$

where local equilibrium between the solid reservoir matrix and the fluid is assumed. The mass source term r_L includes not only well discharge, but also the effects of vertical leakage from less permeable formations adjacent to the main reservoir.

Method of solution

Galerkin method

To solve equations (4) and (5) using a Galerkin-finite element approach (Pinder, Frind 1972) we begin by assuming trial solutions of the form

$$p(x, y, t) \simeq \hat{p}(x, y, t) = \sum_{i=1}^{n} P_i(t)\, u_i(x, y) \qquad (6)$$

for pressure and, for temperature

$$T(x, y, t) \simeq \hat{T}(x, y, t) = \sum_{i=1}^{n} \tau_i(t)\, w_i(x, y) \qquad (7)$$

where $u_i(x, y)$ and $w_i(x, y)$ are basis functions or bases chosen beforehand and satisfying the essential boundary conditions imposed on (4) and (5) respectively. $P_i(t)$ and $\tau_i(t)$ are undetermined coefficients which will be evaluated using linear algebraic methods. The functions $\hat{p}(x, y, t)$ and $\hat{T}(x, y, t)$ will be exact solutions to (4) and (5) respectively only if $L_F(\hat{p}, \hat{T}) = L_T(\hat{p}, \hat{T}) = 0$. This is equivalent to the requirement of the orthogonality of $L_F(\hat{p}, \hat{T})$ to all the basis fuctions $u_i(x, y)$ (i = 1,2,...,n...). Because we have selected only n basis functions, we can impose only n orthogonality conditions and these can be stated as (Kantorovich, Krylov 1964)

$$\iint_D L_F(\hat{p}, \hat{T})\, u_i(x, y)\, dx\, dy = 0 \qquad i = 1,2,\ldots,n \qquad (8)$$

and

$$\iint_D L_T(\hat{p}, \hat{T})\, w_i(x,y)\, dx\, dy = 0 \qquad i = 1,2,\ldots,n \qquad (9)$$

Substituting the trial functions defined by (6) and (7) in (8) and (9), we obtain n equations in n unknowns for each time level, t

$$\iint_D L_F\left(\sum_{j=1}^{n} P_j(t)\, u_j(x,y)\right) \cdot u_i(x,y)\, dx\, dy = 0$$
$$i = 1,2,\ldots,n \qquad (10)$$

and

$$\iint_D L_T\left(\sum_{j=1}^{n} \tau_j(t)\, w_j(x,y)\right) \cdot w_i(x,y)\, dx\, dy = 0$$
$$i = 1,2,\ldots,n \qquad (11)$$

Finite-element integration

The suitability of equations (10) and (11) for solution using linear algebraic techniques depends largely on the choice of the basis functions $u_i(x, y)$ and $w_i(x, y)$. Efficient numerical schemes can be developed when piecewise continuous polynomial functions are used (PRICE ET AL. 1968; PINDER, FRIND 1972). A very powerful approach was introduced by ERGATOUDIS ET AL. (1968) based on higher order polynomial basis functions defined over irregular subspaces (commonly known as finite elements in the structural analysis literature). The basic shape of the element is a quadrilateral, but the sides may be distorted in a prescribed way. The basis functions used in our simulator are modifications of those introduced by ERGATOUDIS ET AL. and permit the use of deformed isoparametric quadrilateral elements of the form presented in Figure 1 and described in detail in PINDER and FRIND (1972). All integrations appearing in (10) and (11) are performed numerically in the local (ξ, η) coordinate system.

Matrix equations

The n equations represented by (10) and (11) may be written in matrix form as

$$[H]\{P\} + [L]\left\{\frac{dP}{dt}\right\} + \{F\} = 0 \qquad (12)$$

for the fluid flow equations, and, for the heat transport equations

$$[M]\{T\} + [N]\left\{\frac{dT}{dt}\right\} + \{R\} = 0 \qquad (13)$$

where H, L, M, and N are square matrices containing coefficients arising out of the integrations appearing in equations (10) and (11), and F and R are column vectors containing known information. The time derivatives appearing in (12) and (13) are approximated using finite difference methods. Although one would anticipate a centered-in-time approximation would provide a more accurate solution than backward difference methods, we have found the converse to be the case. It is important to note from the point of view of computational effort, that while the H, L and N matrices are symmetric, the M matrix is not. As a result the efficient algorithms currently available for solving symmetric, banded matrices cannot be used directly for solving (13).

Wairakei analysis

To demonstrate our ability to simulate single-phase geothermal reservoirs we applied the model outlined above to the Wairakei geothermal field located in the Taupo district on the North Island of New Zealand and illustrated in Figure 2. The Wairakei region is generally

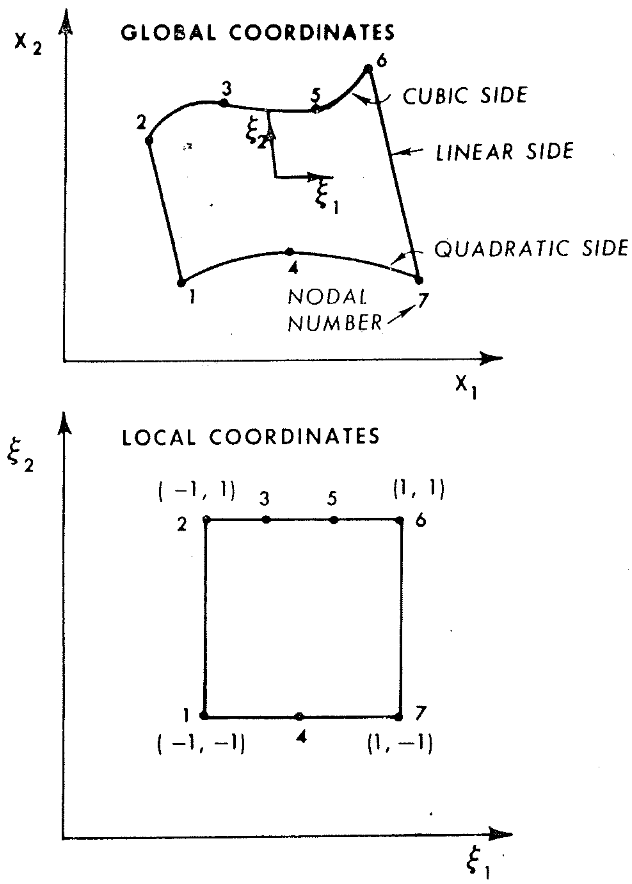

FIG. 1. — *Deformed, mixed, isoparametric quadrilateral element in global x, y and local ξ, η coordinates.*

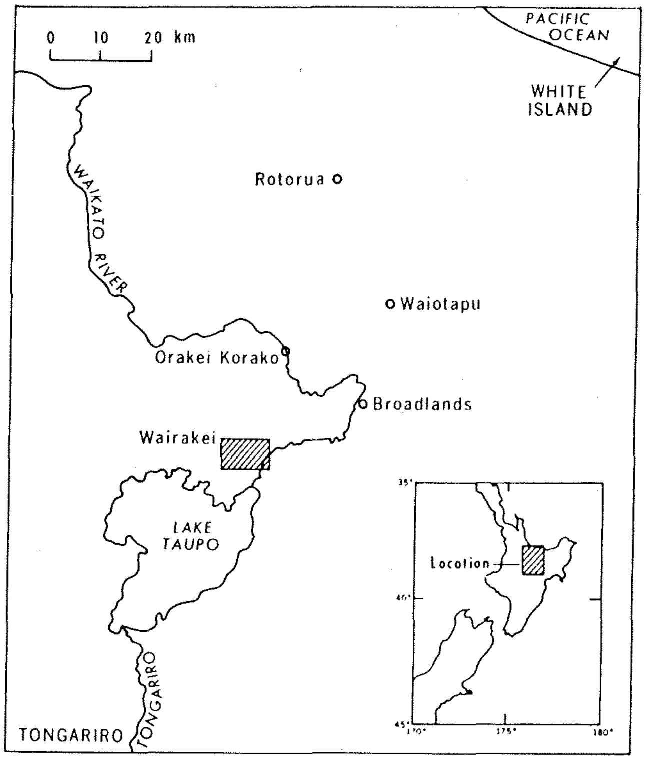

FIG. 2. — *Index map showing location of Wairakei, New Zealand; inset is of North Island.*

considered to occupy a surface area of approximately 15 square kilometers and is centered in an active volcanic belt which appears to be a surface manifestation of a landward extension of the Pacific trench system found north of New Zealand. There is geological evidence to indicate that thermal activity has existed in parts of the Taupo district for the last million years.

Power generation began at Wairakei in 1958 and by 1968 18% of the total electrical requirements of North Island were being met by this facility. Wairakei was the first hot-water system to be developed to drive steam turbines and remained hot-water dominated until approximately 1963 when steam began to form in the reservoir.

GEOHYDROLOGY

The bulk of the steam and hot water discharged by Wairakei bores comes from the Waiora formation which consists mainly of thick pumice breccias and vitric tuffs varying in thickness from about 600 m in the western part of the production zone to more than 800 m in the east. The Waiora formation is overlain by less permeable lacustrine shales of the Huka Falls formation which varies in thickness from 30 to 300 m. The Waiora formation is also underlain by a less permeable formation, the Wairakei ignimbrite. Because of secondary permeability in the form of fractures, however, the base of the Waiora formation does not always appear as a distinct hydrological discontinuity. Although productive bores at Wairakei are nearly always dependent upon zones of locally high permeability associated with major faults, the reservoir as a whole responds as a porous media defined in a continuum sense.

Physical system

The conceptual model on which our simulation is based assumes the existence of a principal producing reservoir, the Waiora formation, overlain by the less permeable Huka Falls formation and underlain by the relatively impermeable Wairakei ignimbrites. We assume that the Huka Falls formation transmits significant quantities of fluid vertically and transmits heat by both conduction and convection. In contrast the Wairakei ignimbrites are assumed impermeable relative to the Waiora formation and heat is therefore transmitted vertically only by conduction.

While many of the physical parameters required to define the Wairakei system are available in the published literature or were obtained through personal communication with colleagues in New Zealand who were familiar with the Wairakei system, other parameters had to be estimated from values obtained for similar hydrologic systems studied elsewhere.

Boundary conditions for fluid flow

The configuration of isoparametric elements used to define the Wairakei geothermal system is presented in Figure 3. Boundary conditions on both temperature and pressure must be defined around the perimeter of the domain defined by this element configuration. Because the Waiora formation is cut off to the south by relatively impermeable rhyolite, this contact is represented in the simulator as a « *no flow* » Neumann condition. The northern and eastern boundaries are also considered impermeable since the pressure responses in bores near these boundaries seem to be independent of the production field. To the west no definitive hydrologic boundary is readily apparent. Accordingly, the western boundary was extended sufficiently far from the production field so as not to influence the transient reservoir performance simulation and was also designated arbitrarily an impermeable boundary. To further justify our choice of boundary conditions for fluid flow, it is interesting to note that this definition of the system is compatible with that deduced from ground subsidence, gravity surveys, and electrical resistivity surveys.

Boundary conditions for energy transport

Inasmuch as temperatures throughout the Wairakei field have changed only slightly with time, constant temperature boundary conditions based on observed temperature data were used. The constant temperatures used in the analysis can be readily deduced through an examination of Figure 4.

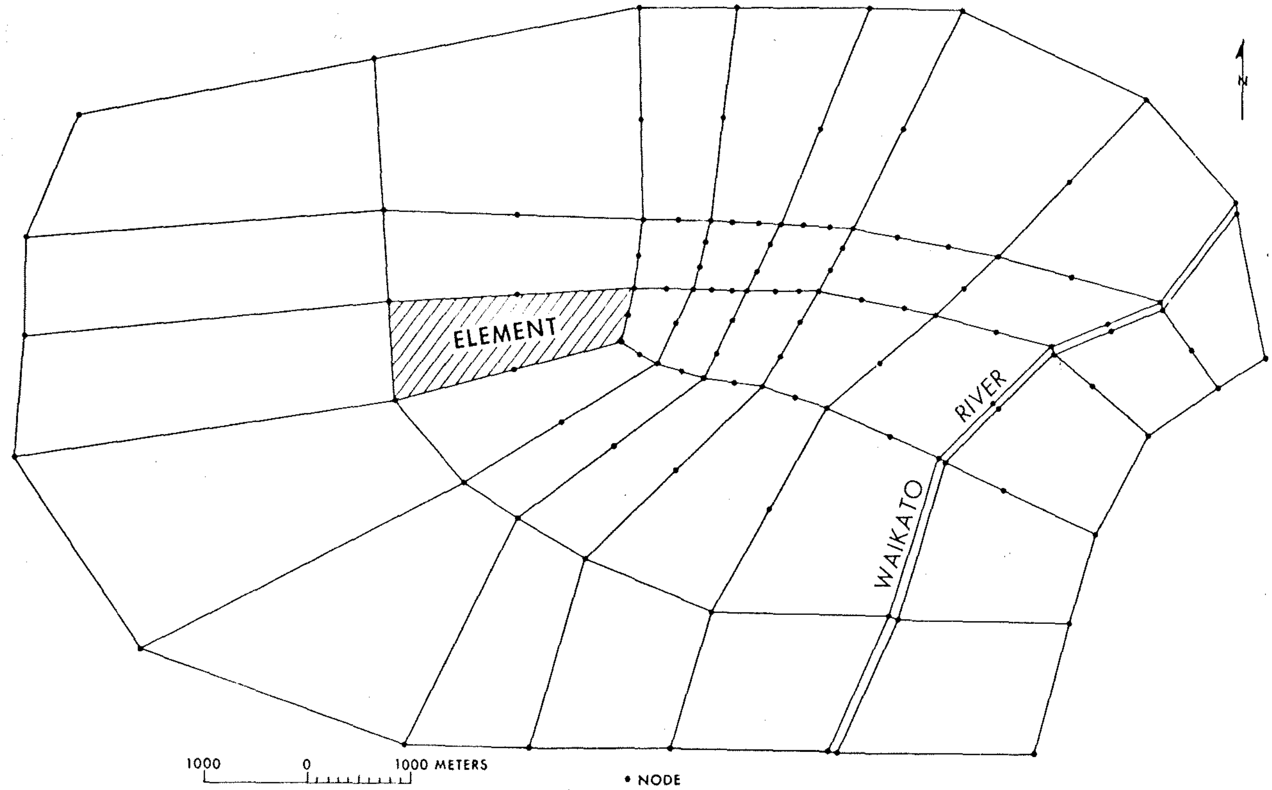

FIG. 3. — *Configuration of isoparametric elements used to define the Wairakei geothermal field for an areal two-dimensional analysis.*

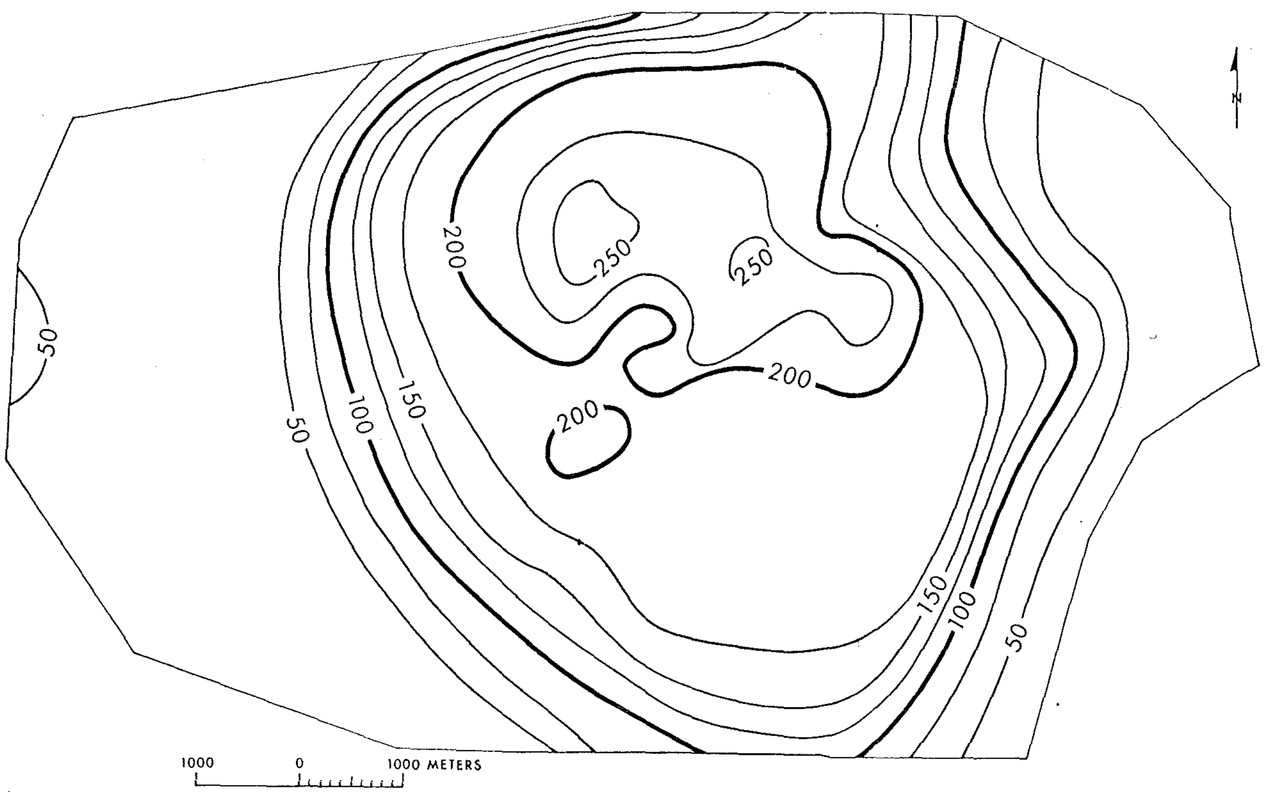

FIG. 4. — *Calculated Wairakei temperature distribution.*

Vertical leakage

The vertical leakage plays an important role in simulating the Wairakei geothermal reservoir. Abundant thermal activity such as steaming ground and hot springs attests to the existence of mass and energy leakage through the Huka Falls formation. To account for this mass and energy transfer a transient one-dimensional solution is generated for vertical flow in this formation. It is assumed that the temperature of fluid entering the reservoir is 12°C, the mean annual air temperature; exiting fluid is assumed to be at reservoir temperature.

Whereas mass and energy leakage at the top of the Waiora formation is relatively easy to simulate, since the problem has been defined fairly accurately through drilling, leakage into the bottom of the reservoir is not as well understood. Consistent with our earlier hypothesis of no vertical mass leakage into the base of the reservoir, all energy transfer at this point is by conduction. To obtain an appropriate functional representation of leakage by conduction from below we assume that prior to exploitation the Wairakei system was at steady-state and, therefore, the amount of energy entering the base of the reservoir by conduction was equal to the energy leaving the top by convection and conduction. Accordingly, the appropriate functional representation was obtained by adjusting the source function until the steady-state heat flow out of the top of the reservoir model matched that measured at the Wairakei field.

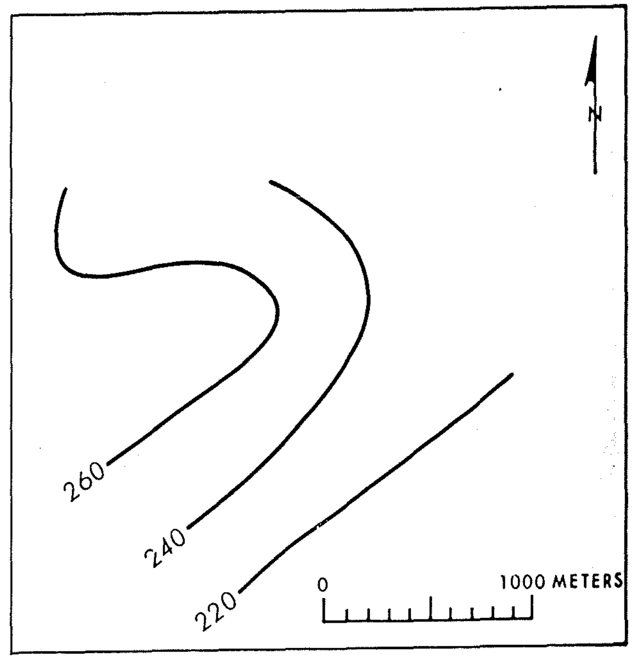

FIG. 5. — *Measured 1955 piezometric surface at Wairakei field, (contour interval 20 m). See Figure 6 for location of contoured area.*

FIG. 6. — *Calculated steady state piezometric surface (contour interval 20 m).*

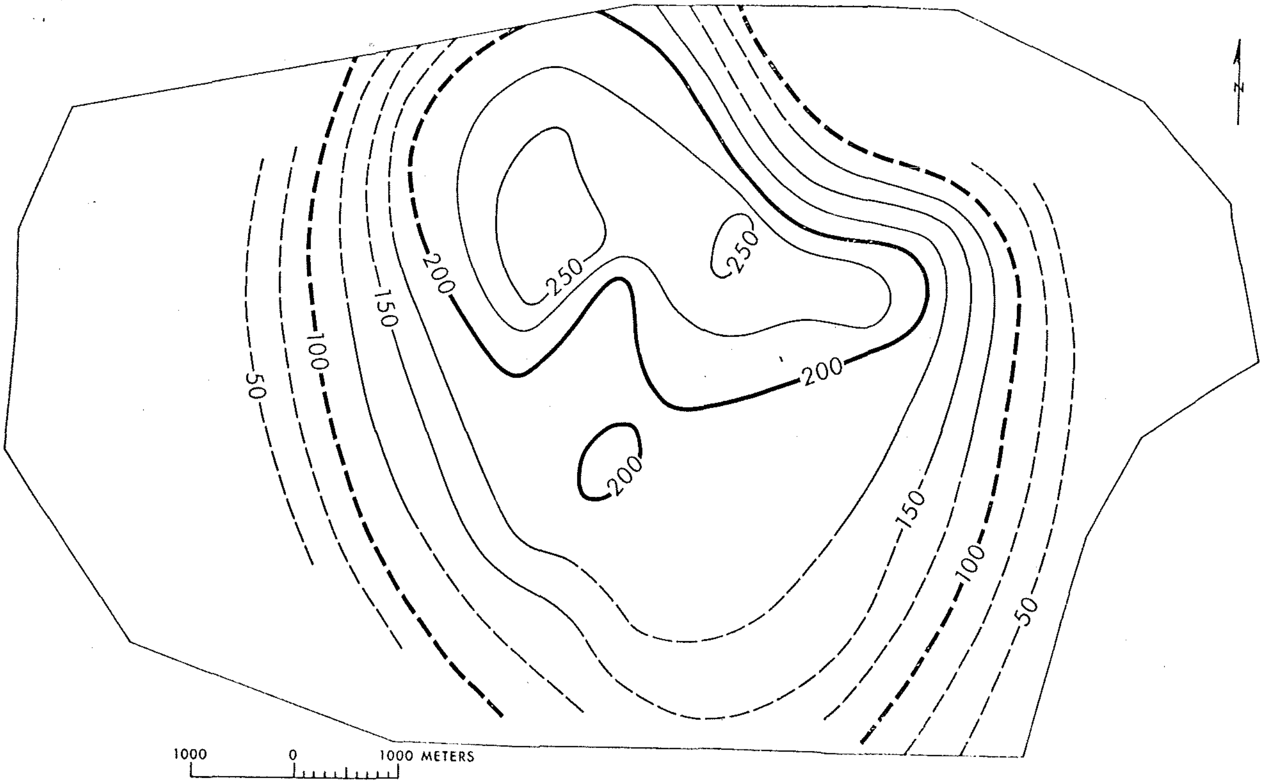

FIG. 7. — *Measured Wairakei steady-state temperature distribution (°C).*

Initial conditions

To obtain the initial conditions for the transient reservoir simulation it was necessary to generate the steady-state conditions observed in bores drilled in the early 1950's prior to significant production. Figure 5 is a piezometric surface of the main production area based on a surface generated by STUDT (1958) and is assumed to represent steady-state conditions. The corresponding steady-state piezometric surface simulated by the numerical model is presented in Figure 6. The rectangular area outlined within the simulated region can be compared directly with Figure 5. Analagous measured and simulated steady-state temperature distributions are presented in Figures 7 and 4 respectively.

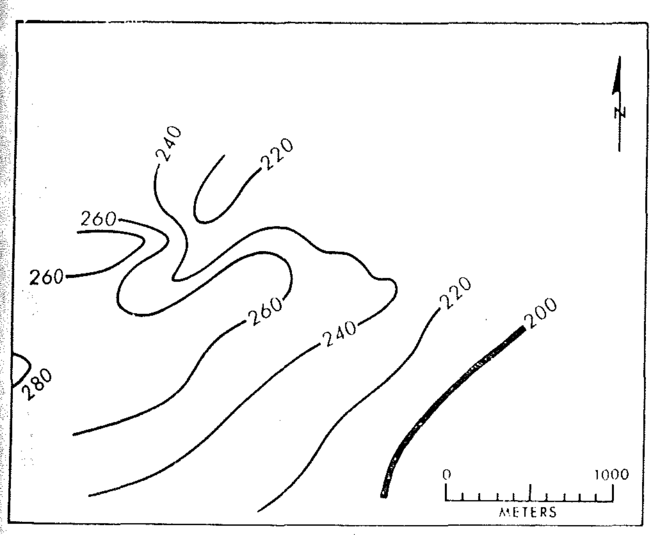

FIG. 8. — *Measured Wairakei piezometric surface, 1958 (after* STUDT *1958). Datum 152.4 m (500 ft) above sea-level (contour interval 20 m).*

SIMULATION

The objectives outlined at the beginning of the paper required simulation not only of the steady-state conditions but also the period of transient response from approximately 1953 to 1963 when the reservoir was being produced for electrical power but remained essentially a hot-water system. To simulate this period the bore discharge data was incorporated into the model and a satisfactory transient analysis was performed using time-step intervals of 30 days. Results from this analysis and measured piezometric surfaces for the years 1958 and 1962 are presented in Figures 8, 9, 10 and 11. Examination of these surfaces shows excellent agreement between computed and observed values of piezometric head. The observed and computed temperature distributions for 1958 and 1962 are approximately the same as presented in Figures 7 and 8 with a maximum decrease of 6°C occuring in the region of maximum drawdown.

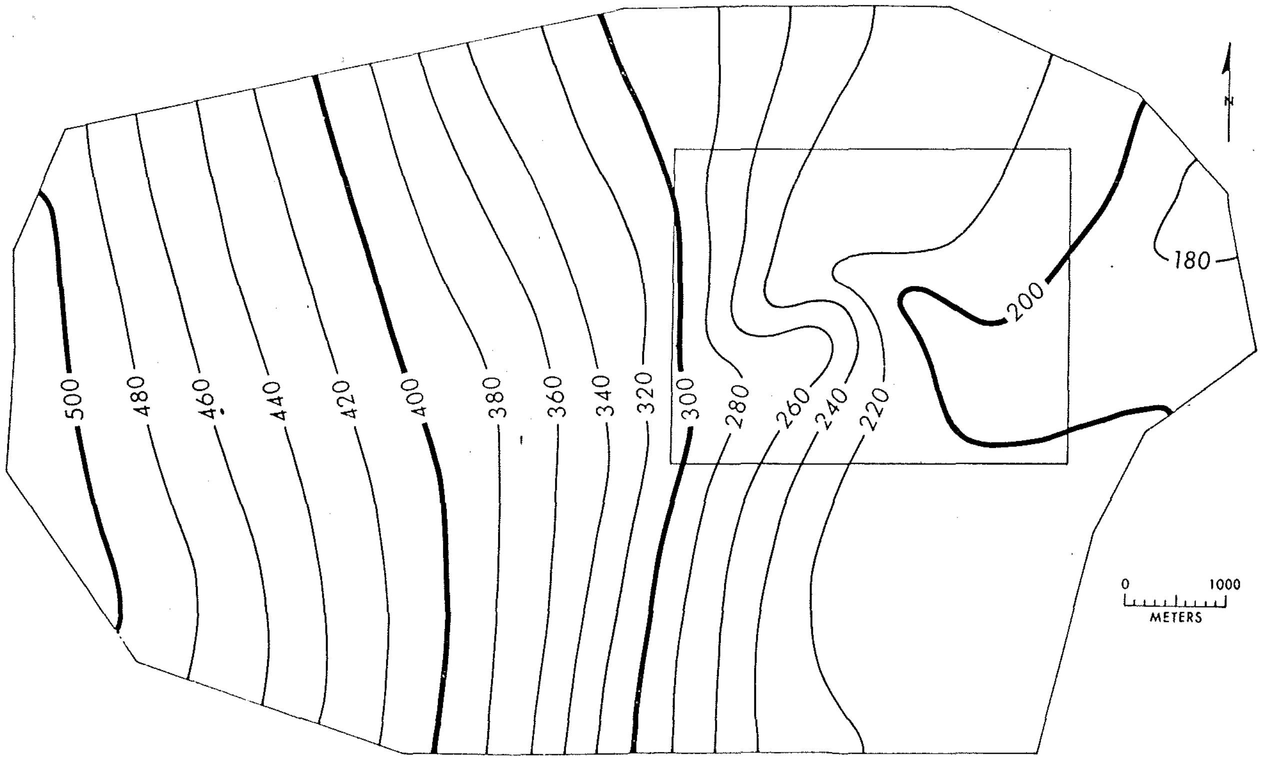

FIG. 9. — *Calculated Wairakei piezometric surface, 1958.*

Notation

- b Saturated thickness of aquifer [L]
- C_v Heat capacity at constant volume, per unit mass $[L^2t^{-2}T^{-1}]$
- $\bar{F}$ External body force per unit mass $[Lt^{-2}]$
- g Gravitational constant $[Lt^{-2}]$
- $\bar{h}$ Heat flow associated with dispersion $[Mt^{-3}]$
- $\bar{\bar{k}}$ Permeability $[L^2]$
- $\bar{\bar{K}}$ Thermal dispersion tensor $[MLt^{-3}T^{-1}]$
- K_d Modular heat diffusion coefficient $[MLt^{-3}T^{-1}]$
- $\bar{m}$ Drag force per unit volume exerted by other phases $[ML^{-2}t^{-2}]$
- p Pressure $[ML^{-1}t^{-2}]$
- P Undetermined coefficient in approximation for pressure $[ML^{-1}t^{-2}]$
- Q Energy source per unit mass $[L^2t^{-3}]$
- r Time rate of supply of mass per unit volume $[ML^{-3}t^{-1}]$
- T Temperature [T]
- U Internal energy per unit mass $[L^2t^{-2}]$
- u Basis function for pressure approximation [dimensionless]
- $\bar{v}$ Velocity $[Lt^{-1}]$
- w Basis function for temperature [dimensionless]
- α Vertical compressibility $[Lt^2M^{-1}]$
- β Coefficient of thermal expansion $[T^{-1}]$
- β_p Compressibility coefficient of the liquid phase $[ML^{-1}t^2]$
- β_T Coefficient of thermal volume expansion for the saturated porous medium $[T^{-1}]$
- ∇_d Unit vector associated with gravity
- θ Porosity [dimensionless]
- k Coefficient of isothermal compressibility $[Lt^2M^{-1}]$
- μ Dynamic viscosity $[ML^{-1}t^{-1}]$
- $\bar{\bar{\pi}}$ Partial stress tensor $[Mt^{-2}L^{-1}]$
- ρ Average density $[ML^{-3}]$
- ρ^* Mass density on a bulk volume basis $[ML^{-3}]$
- τ Undetermined coefficient in approximation for temperature
- $\bar{\bar{\Psi}}$ Porous medium tortuosity tensor [dimensionless]

Subscripts

- ^{o} Symbol refers to standard state
- L Symbol refers to liquid phase
- S Symbol refers to solid phase

Superscripts

- $'$ Symbol refers to source or sink fluid

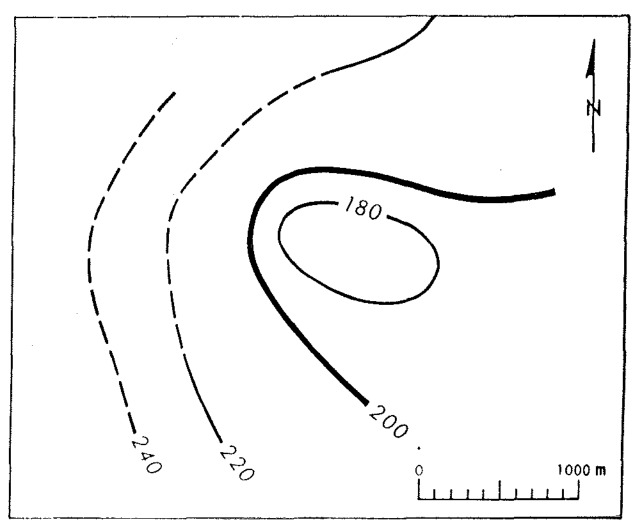

FIG. 10. — *Measured Wairakei piezometric surface, 1962 (data from* GRINDLEY, *1965).*

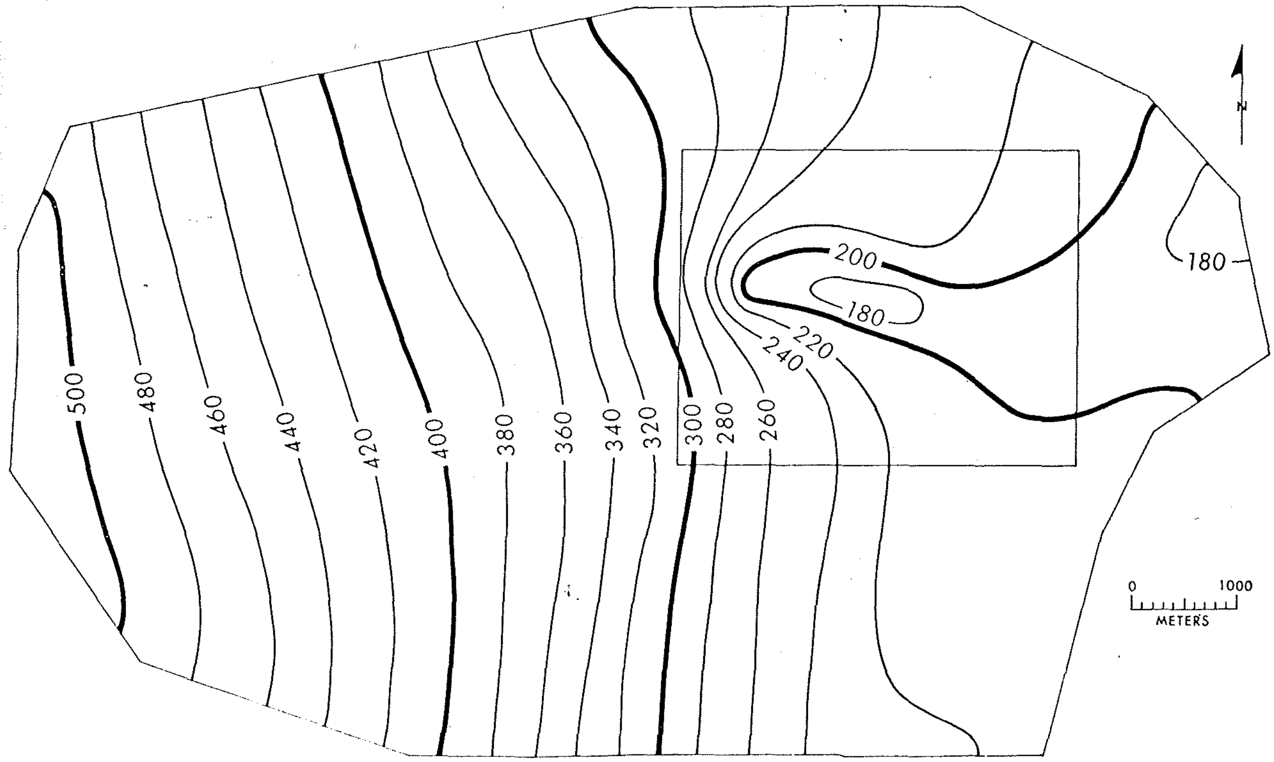

FIG. 11. — *Calculated Wairakei piezometric surface, 1962.*

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