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Inversion of anisotropic magnetotelluric data

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Summary. A geophysical inversion procedure based on the generalized inversion of a matrix is presented and applied to anisotropic magnetotelluric data for one-dimensional models. Various computational aspects of the iterative process involved are discussed together with the use of the resolution matrix, information matrix and the eigenvalues and eigenvectors for establishing the global significance of the parameters and their relationship with the data. A starting anisotropic model is built gradually, the first stage consisting of two separated isotropic problems corresponding to the off-diagonal elements of the impedance-tensor. The examples given include an anisotropic model as well as a model having two anisotropic layers and an isotropic one.

1 Introduction

In this paper we present, discuss and demonstrate a geophysical inversion procedure based on the generalized inversion of a matrix. The process is applied to magnetotelluric data for an anisotropic one-dimensional layered model.

Magnetotelluric field data frequently exhibit anisotropic behaviour. Scalar apparent resistivities defined by Cagniard (1953) often depend upon the orientation of the coordinate system in which the measurements are taken, as well as upon the polarization of the electromagnetic fields. The tensor apparent resistivities (Mann 1965) which are polarization independent similarly depend on the orientation of the coordinate axes used. In order to account for such a behaviour one has to consider more general geo-electrical models than the one-dimensional isotropic one.

A possible extension to models which produce orientation dependent apparent resistivities, is the anisotropic laterally homogeneous one-dimensional model. In such models the conductivity of the various layers is a tensor quantity rather than a scalar one. This is a microscopic property of the composite material in the crust of the Earth. However, considering the averaging-diffusive nature of the very-low-frequency electromagnetic wave propagation in the crust, this microscopic property is a good approximation to a variety of other materials such as rocks with oriented micro-fracturing. In addition, some two-dimensional isotropic models might give field results which are similar to the ones charac-

terizing one-dimensional anisotropic models. This phenomenon is called 'the ambiguity problem' or 'equivalence of the second kind'.

The system function of the Earth's crust in the frequency domain, for the anisotropic model, is the impedance tensor. This is a tensorial extension of the scalar apparent resistivities for the isotropic model and it is computed from the measured vectorial electromagnetic fields (Sims, Bostick & Smith 1971). This complex tensorial function is the reduced field data which contains the information about the anisotropic model. The purpose of the anisotropic magnetotelluric interpretation is to determine the geo-electrical parameters of the model given the values of the impedance tensor elements for various frequencies.

So far the only means for the anisotropic magnetotelluric interpretations has been the forward computation of the impedance tensor for anisotropic models. This problem was treated by several investigators (e.g. O'Brien & Morrison 1967; Loewenthal & Landisman 1973; Abramovici 1974; Shoham & Loewenthal 1976).

The anisotropic magnetotelluric inversion presented here makes use of the analytical expressions for the partial derivatives of the impedance tensor components with respect to the anisotropic model parameters (Abramovici, Landisman & Shoham 1976).

This procedure is based upon the generalized linear inverse theory, thoroughly developed and discussed by Backus & Gilbert (1967, 1968, 1970). They give a variety of seismological examples as well as an approximate treatment of the non-linear problem. A briefer and lucid discussion of the generalized inverse theory is presented by Parker (1971, 1972). For discrete models which are characterized by a finite number of parameters the generalized inverse theory is described in two fundamental works by Jackson (1972) and Wiggins (1972). These works rely upon the generalized inverse of an arbitrary matrix as presented by Lanczos (1958, 1961) and Penrose (1955). In the present discussion it is assumed that the reader is familiar with Jackson's (1972) work.

The linear generalized inversion has been applied to several geophysical problems. Madden (1972) and Jupp & Vozoff (1975) have treated the isotropic magnetotelluric problem. The inversion of vertical magnetic dipole data measured over isotropic model is described by Glenn *et al.* (1973). Inman, Ryu & Ward (1973) use the method to invert direct current resistivity data for one-dimensional isotropic models.

The application of the generalized inversion to the anisotropic magnetotelluric problem presents several advantages. Quite frequently the interpreter is faced by the need to invert noisy data which might be insufficient to completely resolve all the parameters defining a model. This measured data might include some inconsistencies as well as redundancies. Furthermore, the nature of the specific inversion problem might be non-unique. Even in cases where theoretical uniqueness has been established, the inversion of real measured data can exhibit a certain degree of non-uniqueness. The generalized inversion approach enables us to determine some of the model parameters with a finite resolving power from any set of pertinent geophysical data, even for rather ill-posed situations.

Once a proper linearization of the problem is achieved the method seems to suit this delicate task. It is based upon the generalized inverse of an arbitrary matrix which always exists. Consequently the method does not fall apart even for very severe situations. As shown by Jackson (1972) the generalized inversion reduces to the classical least-squares type solution for a well posed case. It continues to work, however, in other cases too, having some desirable stable properties such as giving the solution vector with the smallest norm.

Furthermore, the generalized inversion approach provides the interpreter with some very important and useful means for evaluations:

(a) The information matrix, S (Wiggins 1972) gives the relative importance of the various measurements for the resolution of the specific model, as well as their interdependencies.

(b) The resolution matrix, R (Jackson 1972) shows the relative resolving power of the data set achieved for each of the model's parameters, as well as their interdependencies.

(c) It gives an insight into the nature of the equivalence problem affecting the inversion, through a global formulation for the entire parameter-set.

(d) The double eigensystem (Inman *et al.* 1973) demonstrates which combination of data affects most a specific combination of parameters, and the relative importance of these effects.

(e) It shows the trade-off nature between resolution and accumulation of errors and measures it quantitatively.

All of this very important information can be used for designing the optimal geophysical survey aimed at obtaining the most significant information for achieving a specific target, and for the evaluation of the resolving power of already measured data.

2 Formulation of the problem

The model considered is an anisotropic laterally homogeneous one which consists of a stack of $n-1$ layers overlaying a half space. The conductivity of each layer as well as the conductivity of the half space is a 3×3 symmetric and positive definite tensor. As has been shown by Abramovici (1974) this tensor may be reduced for a laterally homogeneous model to a 2×2 symmetric tensor. Hence the conductivity parameters of the model are $\sigma_{xx}^{(j)}$, $\sigma_{xy}^{(j)}$ and $\sigma_{yy}^{(j)}$, for each layer (Fig. 1). The spatial parameters are defined in the same special coordinate system used by Abramovici *et al.* (1976), with its origin located at the top of the half space and the z axis directed upwards (Fig. 1). Thus, the space parameters are the $n-2$ distances from the origin of each of the interfaces z_j, and the total distance to the free surface H. All the quantities are non-dimensional and scaled in the same manner as described by Abramovici *et al.* (1976). The described $N = 4n-1$ parameters form the parameter vector $\mathbf{x}$ in the following order:

$$\begin{cases} x_{4j+1} = \sigma_{xx}^{(j)};\ x_{4j+2} = \sigma_{xy}^{(j)};\ x_{4j+3} = \sigma_{yy}^{(j)};\ j = 0, 1, \ldots, n-1 \\ x_{4j+4} = z_j;\ j = 1, 2, \ldots, n-2;\ x_4 = H. \end{cases} \tag{2.1}$$

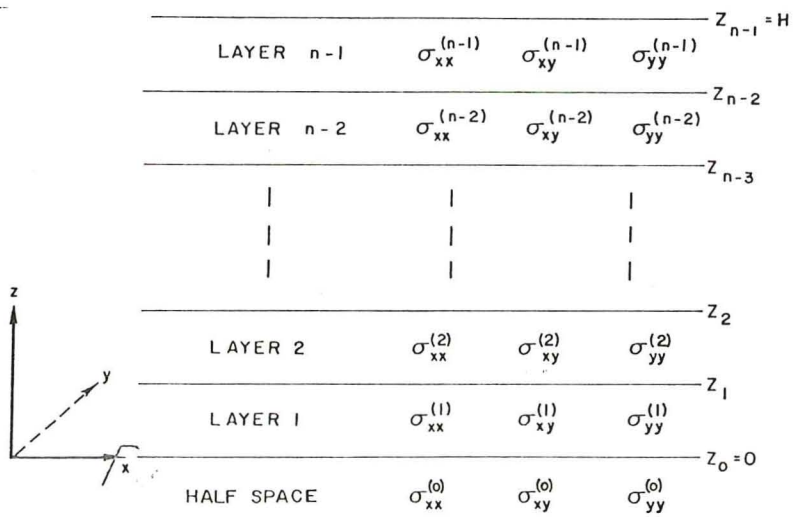

Figure 1. The anisotropic model parameters and the coordinates system.

The reduced magnetotelluric data is the impedance tensor for various frequencies;

$$Z(\omega_i) = \begin{bmatrix} Z_{11}(\omega_i) & Z_{12}(\omega_i) \\ Z_{21}(\omega_i) & Z_{22}(\omega_i) \end{bmatrix}; \quad i = 1, \ldots, l \tag{2.2}$$

For laterally homogeneous models with symmetric conductivity tensors, as is the case here, the impedance tensor components satisfy

$$Z_{11}(\omega_i) + Z_{22}(\omega_i) = 0; \quad i = 1, \ldots, l. \tag{2.3}$$

As demonstrated by several investigators (e.g. Shoham & Loewenthal 1975) equation (2.3) holds true for any anisotropic model with symmetric conductivity tensors regardless of the principal system orientation for each layer.

The diagonal elements can be brought to zero by rotation but the rotation angle is frequency-dependent in general, and consequently the data consists of three complex values or six real values for each frequency. The data can be represented either in a Cartesian form using the real and the imaginary parts of the complex values (which are dependent for a minimum phase function), or in polar form using the tensor apparent resistivities $\rho_{pq}(\omega_i)$ and phases $\phi_{pq}(\omega_i)$

$$\begin{cases} \rho_{pq}(\omega_i) = \dfrac{1}{\omega_i \mu} |Z_{pq}(\omega_i)|^2 \\ \phi_{pq}(\omega_i) = \arctan \dfrac{\mathrm{Im}[Z_{pq}(\omega_i)]}{\mathrm{Re}[Z_{pq}(\omega_i)]}. \\ p, q = 1, 2; \quad i = 1, \ldots, l \end{cases} \tag{2.4}$$

Note that neither ρ_{pq} nor ϕ_{pq} form a tensor. As demonstrated by Abramovici *et al.* (1976), although these are two algebraically equivalent representations they do not necessarily have the same resolving power.

Table 1. The data vector **y** and the four data-modes.

Data \ Mode	(Re, Im)	(ρ, ϕ)	(ρ)	(ϕ)
y_1	$\mathrm{Re}[Z_{11}(\omega_1)]$	$\rho_{11}(\omega_1)$	$\rho_{11}(\omega_1)$	$\phi_{11}(\omega_1)$
y_2	$\mathrm{Re}[Z_{12}(\omega_1)]$	$\rho_{12}(\omega_1)$	$\rho_{12}(\omega_1)$	$\phi_{12}(\omega_2)$
y_3	$\mathrm{Re}[Z_{21}(\omega_1)]$	$\rho_{21}(\omega_1)$	$\rho_{21}(\omega_1)$	$\phi_{21}(\omega_1)$
⋮	⋮	⋮	⋮	⋮
$y_{M/2-2}$	$\mathrm{Re}[Z_{11}(\omega_{M/6})]$	$\rho_{11}(\omega_{M/6})$	$\rho_{11}(\omega_{M/6})$	$\phi_{11}(\omega_{M/6})$
$y_{M/2-1}$	$\mathrm{Re}[Z_{12}(\omega_{M/6})]$	$\rho_{12}(\omega_{M/6})$	$\rho_{12}(\omega_{M/6})$	$\phi_{12}(\omega_{M/6})$
$y_{M/2}$	$\mathrm{Re}[Z_{21}(\omega_{M/6})]$	$\rho_{21}(\omega_{M/6})$	$\rho_{21}(\omega_{M/6})$	$\phi_{21}(\omega_{M/6})$
$y_{M/2+1}$	$\mathrm{Im}[Z_{11}(\omega_1)]$	$\phi_{11}(\omega_1)$	$\rho_{11}(\omega_{M/6+1})$	$\phi_{11}(\omega_{M/6+1})$
$y_{M/2+2}$	$\mathrm{Im}[Z_{12}(\omega_1)]$	$\phi_{12}(\omega_1)$	$\rho_{12}(\omega_{M/6+1})$	$\phi_{12}(\omega_{M/6+1})$
$y_{M/2+3}$	$\mathrm{Im}[Z_{21}(\omega_1)]$	$\phi_{21}(\omega_1)$	$\rho_{21}(\omega_{M/2+1})$	$\phi_{21}(\omega_{M/6+1})$
⋮	⋮	⋮	⋮	⋮
y_{M-2}	$\mathrm{Im}[Z_{11}(\omega_{M/6})]$	$\phi_{11}(\omega_{M/6})$	$\rho_{11}(\omega_{M/3})$	$\phi_{11}(\omega_{M/3})$
y_{M-1}	$\mathrm{Im}[Z_{12}(\omega_{M/6})]$	$\phi_{12}(\omega_{M/6})$	$\rho_{12}(\omega_{M/3})$	$\phi_{12}(\omega_{M/3})$
y_M	$\mathrm{Im}[Z_{21}(\omega_{M/6})]$	$\phi_{21}(\omega_{M/6})$	$\rho_{21}(\omega_{M/3})$	$\phi_{21}(\omega_{M/3})$

Dealing with field data it is well known that the apparent resistivities are noisier than the phases for some sites and the opposite may be true in other occasions. Since we want to enable the magnetotelluric interpreter to use the data which is the quietest and which resolves the desired model best, the inversion has been developed and programmed to operate in four data-modes. The data forms an M-component real data vector $\mathbf{y}$. The data-modes and the order of the data vector components are summarized in Table 1.

The data vector components y_i are non-linear functions of the parameter vector components x_j

$$y_i = y_i(x_1, \ldots, x_N); \quad i = 1, \ldots, M. \tag{2.5}$$

Following Jackson (1972) the forward problem can be linearized using Taylor expansion and ignoring second and higher order terms, about some initial parameter vector $\mathbf{x}_0$. Doing so we get a linear relationship

$$\Delta \mathbf{y} = A \cdot \Delta \mathbf{x} \tag{2.6}$$

where A is an $M \times N$ matrix with elements

$$A_{ij} = \left.\frac{\partial y_i}{\partial x_j}\right|_{\mathbf{x}_0} \quad \begin{matrix} i = 1, \ldots, M \\ j = 1, \ldots, N \end{matrix} \tag{2.7}$$

and

$$\begin{cases} \Delta \mathbf{x} = \mathbf{x} - \mathbf{x}_0 \\ \Delta \mathbf{y} = \mathbf{y} - \mathbf{y}(\mathbf{x}_0). \end{cases} \tag{2.8}$$

The elements of the matrix A are the first order partial derivatives of the data computed at the initial parameter set $\mathbf{x}_0$. Using the expressions for the derivatives developed by Abramovici *et al.* (1976), we get the derivatives for the first, Cartesian, data mode. In order to compute the first order partial derivatives for the other, polar, data modes the following relations are used

$$\frac{\partial \rho_{pq}}{\partial x_j} = \frac{2}{\omega \mu} \left(\mathrm{Re}[Z_{pq}] \cdot \mathrm{Re} \frac{\partial Z_{pq}}{\partial x_j} + \mathrm{Im}[Z_{pq}] \cdot \mathrm{Im} \frac{\partial Z_{pq}}{\partial x_j} \right) \tag{2.9}$$

$$\frac{\partial \phi_{pq}}{\partial x_j} = \frac{1}{|Z_{pq}|^2} \left(\mathrm{Re}[Z_{pq}] \cdot \mathrm{Im} \frac{\partial Z_{pq}}{\partial x_j} - \mathrm{Im}[Z_{pq}] \cdot \mathrm{Re} \frac{\partial Z_{pq}}{\partial x_j} \right). \tag{2.10}$$

3 Iterative generalized inversion

The general first-order approximation equation (2.6) is a system of linear equations relating the generally unknown parameter-differences vector $\Delta \mathbf{x}$ to the generally known data differences vector $\Delta \mathbf{y}$. If the functional relation (2.5) between the parameters and the data were linear, noise free, $M = N$ and the matrix A non-singular, the problem could be readily solved by multiplying both sides of equation (2.5) by the inverse A^{-1}. In general these conditions are not satisfied for the problem we want to treat here. M is not necessarily equal to N, commonly we have $M > N$ which means that the system is overconstrained and might be inconsistent. The rank p of the system matrix may be smaller than min $\{M, N\}$, i.e. the system is underdetermined at the same time. The data we have contain some noise and the relation (2.5) for magnetotellurics is non-linear. Consequently, one has to use something different than the usual inverse matrix A^{-1}, which does not exist here, such as the generalized

inverse of a matrix (Lanczos 1958, 1961) which is denoted by H

$$H = V\Lambda^{-1}U^T \tag{3.1}$$

Λ^{-1} is the inverse of the diagonal matrix Λ having as diagonal elements the q ordered largest common non-zero eigenvalues of A^TA and AA^T. The columns of V are the corresponding q Nth order eigenvectors of A^TA while U is similarly constructed using the corresponding q Mth order eigenvectors of AA^T. Clearly q cannot be larger than p.

Using this unique generalized inverse, a unique estimate of the unknown parameter differences vector can be computed

$$\hat{\Delta\mathbf{x}} = H\,\Delta\mathbf{y}. \tag{3.2}$$

This equation is used to define an iterative process in order to successively improve on an initial model parameter vector $\mathbf{x}_1$

$$\hat{\Delta\mathbf{x}}_i = H\,\Delta\mathbf{y}_i; \qquad i = 1, 2, \ldots, \tag{3.3}$$

$$\begin{cases} \hat{\Delta\mathbf{x}}_i = \mathbf{x}_{i+1} - \mathbf{x}_i \\ \Delta\mathbf{y}_i = \mathbf{y}_0 - \mathbf{y}_i \end{cases}; \qquad i = 1, 2, \ldots \tag{3.4}$$

$\mathbf{y}_0$ is the measured data vector. The process is continued until the relative L_2 norm satisfies

$$\frac{\|\hat{\Delta\mathbf{x}}_i\|}{\|\mathbf{x}_i\|} < \epsilon. \tag{3.5}$$

The quality of fit is determined from

$$\frac{\|\Delta\mathbf{y}_i\|}{\|\mathbf{y}_i\|} < \eta. \tag{3.6}$$

Both of these criteria are needed in order to evaluate the process. While (3.5) is used here as a termination criterion, (3.6) serves as a quality measure. It should be noted that since our problem is non-linear, the final model might depend on the initial one.

Combining equations (2.6) and (3.3) we get

$$\hat{\Delta\mathbf{x}} = HA\ \Delta\mathbf{x} \equiv R\ \Delta\mathbf{x} \tag{3.7}$$

R is called the resolution matrix (Jackson 1972). Its rows are cross-correlation windows through which the exact solution space is mapped into the unique estimate $\hat{\Delta\mathbf{x}}$. The closer R is to the identity matrix of order N the better resolution in $\hat{\Delta\mathbf{x}}$ we get from the data.

Equation (3.7) can be interpreted in terms of the equivalence phenomenon. Since the rows of R determine the relative contributions of $\Delta\mathbf{x}$ elements to the estimate of each specific component of $\hat{\Delta\mathbf{x}}$ they actually represent parameters interdependencies. These interdependencies are responsible for the equivalence phenomenon in magnetotelluric soundings, as well as in direct current soundings. It follows that R provides, through equation (3.7) a general way to represent *globally* the equivalence family of a certain parameter set representing a geo-electrical model. Usually this term is used in connection with parameters which belong to the same layer, which is a partial and special case of the global equivalence described in (3.7).

A certain number, q of eigenvalues and eigenvectors is used to compute H in each iterative step. In many cases this number should be taken smaller than its maximum value p. The reason is to avoid excessive amplification of data noise due to some possibly small eigenvalues which cause large variance in $\hat{\Delta\mathbf{x}}$ and inaccuracies caused by numerical errors in the computation of these small eigenvalues. This number q, called the effective number of

degrees of freedom, may be optimally determined using the following approaches: (1) If one has a good idea about the variances in the data, a trade-off between the global variance of the problem and the deviation of R from the identity matrix, both as a function of q is used (Jackson 1972; Abramovici *et al.* 1976). (2) A threshold for the minimal eigenvalue is set, and only q eigenvalues with absolute values (normalized by the largest eigenvalue) above this threshold are considered in computing the generalized inverse.

The resolution matrix deviates from the identity matrix as q gets smaller. Consequently q, or the relative effective number of degrees of freedom q/N, is an appropriate relative measure of the degree of equivalence affecting a generalized inversion stage which relates specific data to a certain parameter set. Left multiplying equation (3.2) by A we get

$$\hat{\Delta \mathbf{y}} \equiv A\,\hat{\Delta \mathbf{x}} = AH\,\Delta \mathbf{y} \equiv S\,\Delta \mathbf{y} \tag{3.8}$$

S is the information matrix (Wiggins 1972). The rows of S are cross-correlation windows through which we get the elements of the estimated $\hat{\Delta \mathbf{y}}$ defined here, from the data difference vector $\Delta \mathbf{y}$. S deviates from the Mth order identity matrix as q gets smaller. Obviously if $M > N$, S cannot be an identity matrix. Interdependencies in the data are represented by the rows of S. Thus by examining S it is possible to find groups of data which support each other. The magnitudes of the elements of S reflect the relative importance of the corresponding data in the inversion and therefore S can assist in deciding which data are essential and which convey relatively less important information concerning a certain parameter set, representing a geoelectrical model.

An initial model is required in order to start the iterative process defined in equations (3.3) and (3.4). As was mentioned before, the final model might depend on the initial one.

We treat the anisotropic first guess problem by approximately separating it into a couple of isotropic similar ones. This approach has two major advantages. First, the isotropic first guess problem is relatively simpler, involving less parameters to guess. Second, in most real situations the two scalar isotropic problems are not totally independent since at least some of the interfaces share the same z coordinate value. Therefore we can infer from one first guess to another, improving the quality of both of them.

The raw data for the anisotropic first guess is the unrotated impedance tensor as function of frequency, expressed in the measuring coordinates, equation (2.2). The elements of this tensor satisfy equations (2.3). For each frequency there is a coordinate system, in which both diagonal elements of the impedance tensor vanish. This frequency-dependent system of coordinates is called the principal system and is related to the measuring coordinates through a rotation in the x–y plane. Since equation (2.3) is satisfied also by the magnetotelluric impedance tensor for a two-dimensional isotropic model, we shall analogously call the direction of one of the axes the strike. This strike direction is frequency dependent unless there is a coordinate system in which all the model's conductivity tensors are diagonal, which is a special case for which the anisotropic problem can be completely separated into two isotropic ones. In the general case an optimal, global strike direction can be found for the entire data set by finding a rotation angle α to satisfy

$$\Psi(\alpha) \equiv \sum_{i=1}^{I} |Z_{11}(\omega_i, \alpha)|^2 = \text{minimum}. \tag{3.9}$$

For the special case mentioned above this minimum is zero. Applying rotation transformations to the impedance tensor, differentiating $\Psi(\alpha)$ and using the necessary condition for a minimum

$$\frac{\partial \Psi(\alpha)}{\partial \alpha} = 0 \tag{3.10}$$

we get an expression for this optimal rotation angle

$$\alpha = \tfrac{1}{4} \arctan \frac{\left(\left[\sum_{j=1}^{l} [R_{11}(R_{12}+R_{21}) + I_{11}(I_{12}+I_{21})] \right] \right.}{\left. \left[\sum_{i=1}^{l} \{\tfrac{1}{4}[(R_{12}+R_{21})^2 + (I_{12}+I_{21})^2] - (R_{11}^2 + I_{11}^2)\} \right] \right)} \tag{3.11}$$

where

$$\begin{cases} R_{pq} = \mathrm{Re}\,\{Z_{pq}(\omega_i, 0)\} \\ I_{pq} = \mathrm{Im}\,\{Z_{pq}(\omega_i, 0)\} \\ i = 1, \ldots, l; \quad p, q = 1, 2. \end{cases} \tag{3.12}$$

The raw data is rotated through an angle α to this optimal global principal system. The generalized inversion process treated here is carried out entirely in this system using the transformed impedance tensor as data.

This rotation, based on minimizing $\Psi(\alpha)$ involving $Z_{11}(\omega, \alpha)$ only, is suitable for treating data which are known to represent a model which belongs to the class considered here. Dealing with general field data it may be necessary to include both diagonal elements of the impedance tensor in the definition of $\Psi(\alpha)$, equation (3.9). If the data represent a different kind of model, this generalized inversion procedure will yield an anisotropic model that best approximates the field data in the least-squares sense.

Once the global principal system is found and the data transformed to it, we consider two separated isotropic problems with apparent resistivities and phases computed from the two off-diagonal impedance-tensor elements using equation (2.4). Thus the next stage is to find two isotropic initial models, e.g. by using the method described by Patrick & Bostick (1969) or through an interactive simple and fast isotropic forward modelling program employing trial and error. The initial guess may be further refined using an isotropic inversion. Such a method is described by Wu (1968), Patrick & Bostick (1969) and others. We preferred to use an isotropic, generalized inversion procedure. This refinement is optional, and has to be carried out only if a close enough first guess could not be found using the direct and faster means. The two initial isotropic models are merged now into a single anisotropic model, in which all the conductivity tensors are diagonal. All the interfaces from both models are considered. Thus, if no interfaces coincide, the number of layers in this model is equal to the sum of the numbers of layers in each of the two isotropic models and in order to avoid electrically insignificant layers or interfaces, some of the layers are combined with adjacent ones. The model obtained is taken as the initial anisotropic model.

This model cannot account for the diagonal impedance tensor elements. However, these have a minimum value in the global principal system and therefore the initial model may be close enough. The logical steps followed in order to get the anisotropic initial model are summarized in Fig. 2.

The next stage consists of K cycles. In each cycle the data tensor has diagonal elements becoming gradually closer to the given ones;

$$\begin{bmatrix} \frac{k}{K} Z_{11} & Z_{12} \\ Z_{21} & \frac{k}{K} Z_{22} \end{bmatrix}; \quad k = 1, \ldots, K. \tag{3.13}$$

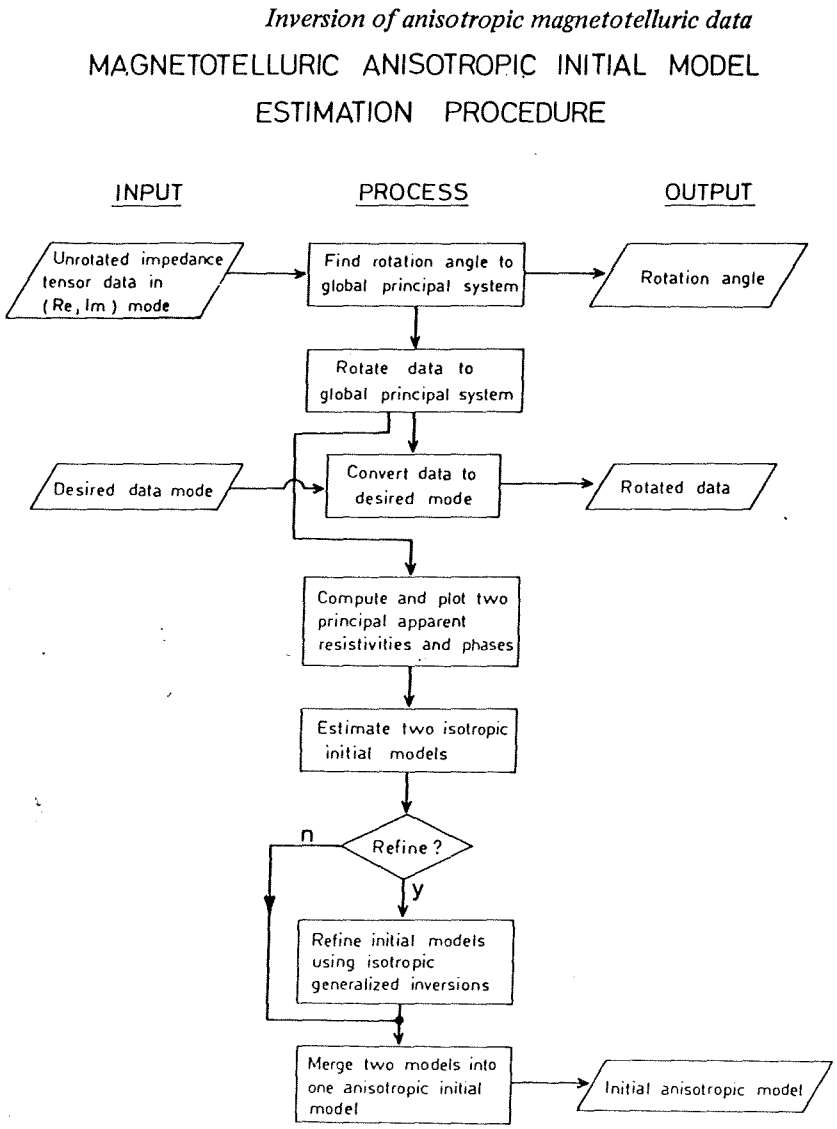

Figure 2. Simplified block diagram for initial anisotropic model estimation.

The final model obtained after the process converges in step k is used as an initial one for the step $k + 1$. After the last cycle is completed we get the inverted anisotropic model.

4 Computational and operational considerations

A computer program which executes the anisotropic generalized inversion was written. The Tel-Aviv University CDC-6600 computer was utilized to develop the program and for all the computational examples presented in the next section. Special care has been taken to carry out the calculations as accurately as possible. All the forward computations of the impedance tensor elements and their various derivatives were done using the analytical expressions given by Abramovici (1974) and Abramovici *et al.* (1976), assuring the high accuracy which is essential for the derivative-sensitive generalized inversion procedure. A crucial stage is the computation of the generalized inverse matrix H of equation (3.1)

which involves the solution of two eigensystems in order to compute all the eigenvalues and eigenvectors of AA^T and A^TA. We have examined several procedures and decided to use an approach recommended, presented and programmed by Smith *et al.* (1974) in their EISPACK subroutine package which we have found to be the most accurate.

The computer program can be used in either of two basic operational modes, a forward resolution analysis mode and an iterative inversion mode. Each of these modes has a variety of options enabling the interpreter to benefit most from the inversion, to optimize it according to a specific data set, parameter set and purpose, and to obtain all the pertinent information concerning the process and the model which the process can yield.

In the forward resolution analysis operational mode, the geoelectrical parameters of the initial model are used to form the parameter vector $\mathbf{x}$ according to equation (2.1). A forward computation is then used to calculate the data vector $\mathbf{y}$ for the given set of frequencies. The program can be terminated at this point if one wishes merely to do forward computation. Next, all the derivatives are computed to form the system matric A, equation (2.7). Using this matrix the eigenvalues and eigenvectors of AA^T and A^TA are found and presented graphically in a fashion similar to the one used by Inman *et al.* (1973). A desired number of resolution and information matrices pairs for various numbers of effective degrees of freedom q, beginning with its maximum value and going down, is computed. All the matrices are normalized to the open interval $(-10; 10)$ and printed, leaving a blank wherever a null value occurs for the sake of clarity of presentation.

This forward resolution analysis is especially valuable if one has some idea concerning the model, in order to evaluate the feasibility and the resolving power of a magnetotelluric survey for such a model, and in order to optimally design this survey. It is also very helpful in the evaluation of a model obtained in the iterative mode. The resolution matrix for the effective number q of the degrees of freedom used in most iterations can reveal the relative resolution in each of the parameters and the equivalence families affecting the inverted model. The information matrix for this q describes the relative importance of the data and their interdependencies. The coupled and ordered eigensystems provide a graphical presentation from which one can tell which combination of data contributes most to the determination of a specific parameter combination. An idea concerning the noise effect on specific parameters may be obtained from the order of this eigensystem and the relative magnitudes of the eigenvalues.

Any number of the following three options may be used in the forward resolution analysis mode:

(1) Any of the data-modes as shown in Table 1.

(2) Any number of parameters may be fixed. A fixed parameter is removed from the parameter vector and the corresponding derivatives are not included in the system matrix A. This parameter is considered known and is ignored by the process.

(3) A transformed set of parameters may be used in which the parameters representing the diagonal values of the conductivity tensors and the vertical coordinate parameters are replaced by their natural logarithms. The corresponding derivatives are transformed accordingly.

In the iterative operational mode an initial parameter vector $\mathbf{x}_1$ is formed using the parameters of the first guess model. The data vector $\mathbf{y}_0$ is formed using the impedance tensor data rotated to the global principal system. K cycles of iterative generalized inversion are performed as described in the previous section. The termination criterion defined in equation (3.5) is employed in each cycle using a desired ϵ. The maximum allowed number of iterations in each cycle may be externally set by the user. For evaluation purposes $\|\Delta\hat{\mathbf{x}}_i\|$, $\|\mathbf{x}_i\|$ and $\|\Delta\mathbf{y}_i\|$ are printed for each iteration as well as $\mathbf{x}_i$, $\Delta\mathbf{x}_i$, $\mathbf{y}_i$ and $\Delta\mathbf{y}_i$.

After all K iterative inversion cycles are completed and a final model is found, a forward resolution analysis is performed for this model and the given frequencies. If any of these iterative cycles fails to converge – a forward resolution analysis is performed on the initial model and the user may benefit from the information provided about this model.

All the three options described for the forward analysis mode may be used in the iterative mode as well. The parameter fixing option is very useful if one has some knowledge about the model from other independent sources. In the final model the value of a parameter which has been fixed is identical to its value in the initial one. The transformed logarithmic parameters option is a way to guarantee the positiveness of those parameters which should not become negative (Patrick & Bostick 1969).

In the iterative mode, any of the following additional options may be used:

(1) An acceptance test which is performed on the parameter vector after each iteration. This test checks for the positiveness of the diagonal elements of each conductivity tensor and of the layer thicknesses as well as whether each of the conductivity tensors is positive definite.

(2) The parameter-variations vector elements, $\Delta\hat{\mathbf{x}}_i$ may be bounded such that the conditions of the acceptance test described above are not violated.

Clearly, some of the operational options available in the iterative mode are not independent. However, the ability to employ any of them, choosing the ones which are most suitable for the specific inversion problem, gives the user considerable flexibility and convenience in applying this anisotropic generalized inversion procedure. The main logical steps of the anisotropic generalized inversion program are described in Fig. 3.

5 Computational examples

In order to check the anisotropic generalized inversion algorithm and the computational procedure, we applied them to a number of models, most of them having all the layers anisotropic, and some composed of both isotropic and anisotropic layers. All the options which have been described in the previous sections and in Fig. 3 have been tested.

The procedure and the computer program can be used in a multitude of ways, depending on the various selections made. Obviously we cannot demonstrate all the possibilities here and we shall have to restrict ourselves to some characteristic examples, in order to aid possible users of the method.

The examples shown here are synthetic. In order to further check the method, the synthetic data, taken as measured data, were produced using the method of Shoham & Loewenthal (1975) which is independent of the forward computation by Abramovici (1974), used in the work presented here. In the algorithm by Shoham & Loewenthal (1975) the symmetric conductivity tensor of each layer is expressed by its two diagonal elements, the principal conductivities, $\sigma_i^{(1)}$, $\sigma_i^{(2)}$, and the angle θ_i between the measuring coordinates and the principal system of each layer. The synthetic data which is computed by using this algorithm is expressed in the measuring coordinates.

Although the examples we present here are synthetic, they are not unrealistic. In fact the parameters of the models treated here were chosen such that they give rotated principal apparent resistivities which are similar to both TE and TM modes measured at the Zohar magnetotelluric field site (Shoham, Ginzburg & Abramovici, in preparation).

5.1 EXAMPLE 1

Model A which is considered in the first example consists of three anisotropic layers over an

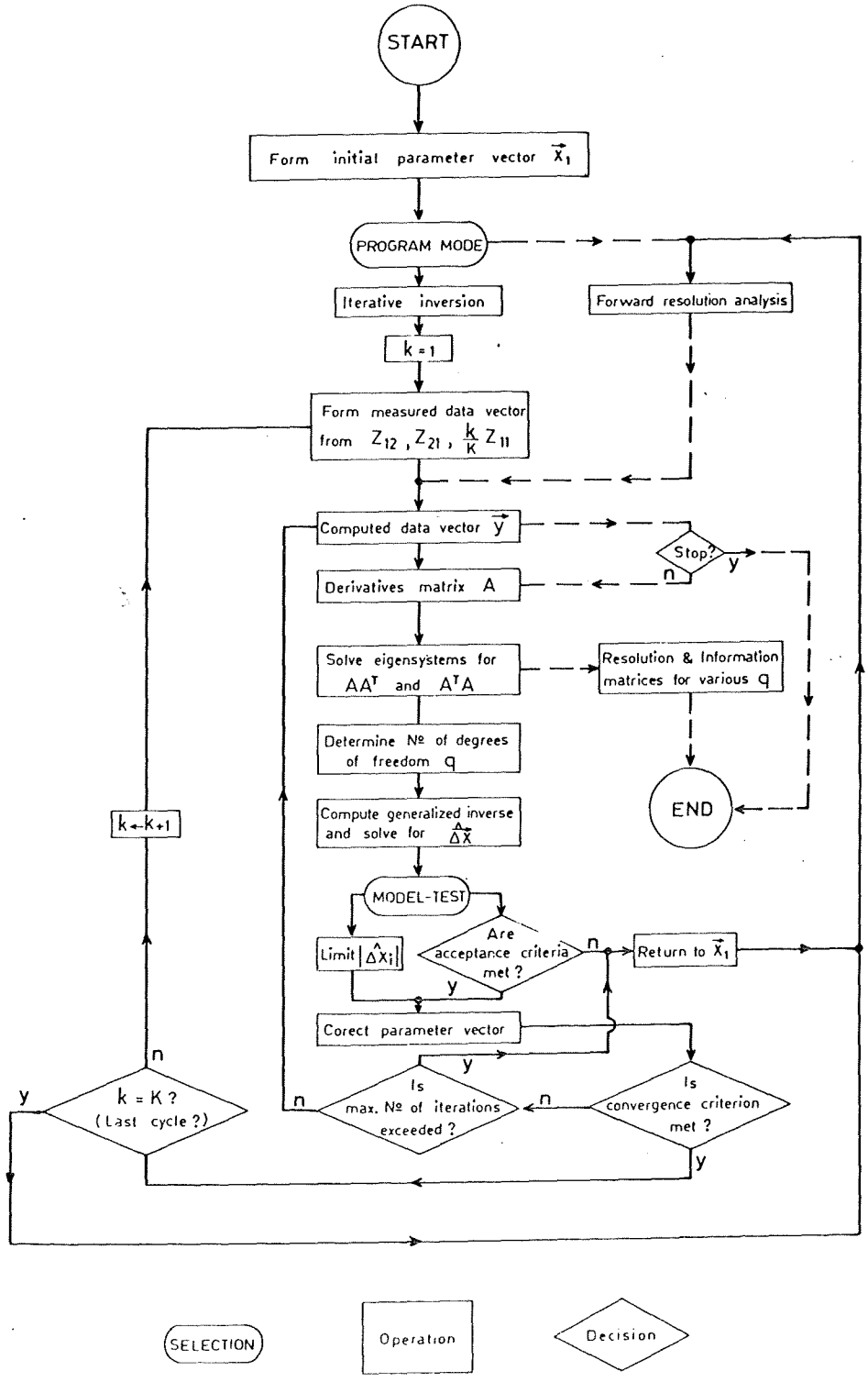

Figure 3. Simplified block diagram for anisotropic generalized inversion.

Table 2. Parameters for model A, related to the measuring coordinates.

Layer	z_i (m)	Principal conductivities $\sigma_i^{(1)}$	$\sigma_i^{(2)}$	θ_1 (deg)
HS	0.0	1.000	2.000×10^{-1}	6.0
1	20000.0	2.560×10^{-2}	1.563×10^{-2}	20.0
2	22000.0	2.025×10^{-1}	1.225×10^{-1}	30.0
3	23000.0	9.000×10^{-2}	4.000×10^{-2}	5.0

anisotropic half-space. The parameters of this model which were used to compute the synthetic data are given in Table 2. The conductivity values in this table as well as throughout this work are in $(\Omega m)^{-1}$.

In the first stage the impedance tensor elements as function of frequency are computed in the measuring coordinate system. Next a rotation angle of -10.221 degrees is found as described in Section 3 using equation (3.11). The impedance tensor data are rotated through this angle into the global principal system. This is the system in which we shall perform the generalized inversion, and in which the global contribution of the diagonal elements of the impedance tensor is minimized. The parameters of model A expressed in this principal system are given in Table 3.

In the next step a generalized inversion is performed. The data for the inversion consists of the impedance tensor elements *in* (Re, Im) mode for ten frequencies, equilogarithmically spaced: $\omega_1 = 25.119$ cps and $\omega_{10} = 10^{-4}$ cps. This is a common frequency range in magnetotelluric surveying. The order of the data vector elements is shown in Table 1. The generalized inversion is performed in two inversion cycles, i.e. $K = 2$. The termination criterion of equation (3.5) is used with $\epsilon = 10^{-5}$. For demonstration purposes one parameter, x_4, is fixed on its true value. As seen from equation (2.1) this parameter represents the total thickness H of the stack of layers over the half space.

Three initial models were tried for this example, and for all of them the generalized inversion process converged yielding the same final model. The parameters of the first initial model are given in Table 4. These parameters are the ones of Table 3 perturbed by some ± 25 per cent. The second initial model is equal to the first one except for $\sigma_{12}^{(j)}$, $i = 0, \ldots, 3$, which are equal to zero. This second initial model represents one which is obtained by the procedure described in Fig. 2.

Using the initial model as shown in Table 4, the first generalized inversion cycle converged after seven iterations, and the second cycle took three iterations to converge.

Table 3. Model A parameters expressed in the global principal system.

Layer	z_i(m)	$\sigma_{11}^{(i)}$	$\sigma_{12}^{(i)}$	$\sigma_{22}^{(i)}$
HS	0.0	9.957×10^{-1}	-5.872×10^{-2}	2.043×10^{-1}
1	20000.0	2.531×10^{-2}	1.670×10^{-3}	1.591×10^{-2}
2	22000.0	1.933×10^{-1}	2.547×10^{-2}	1.317×10^{-1}
3	23000.0	8.959×10^{-2}	-4.531×10^{-4}	4.041×10^{-2}

Table 4. First set of initial parameters for model A inversion.

Layer	z_i(m)	$\sigma_{11}^{(i)}$	$\sigma_{12}^{(i)}$	$\sigma_{22}^{(i)}$
HS	0.0	1.240	1.040×10^{-1}	2.600×10^{-1}
1	19000.0	1.900×10^{-2}	2.500×10^{-3}	1.300×10^{-2}
2	21500.0	2.300×10^{-1}	2.600×10^{-2}	1.800×10^{-1}
3	23000.0	7.500×10^{-2}	5.400×10^{-3}	3.000×10^{-2}

Table 5. Third set of initial parameters for model A inversion.

Layer	z_i (m)	$\sigma_{11}^{(i)}$	$\sigma_{12}^{(i)}$	$\sigma_{22}^{(i)}$
HS	0.0	2.000×10^{-2}	2.000×10^{-2}	4.000×10^{-1}
1	17000.0	1.200×10^{-2}	1.300×10^{-3}	8.000×10^{-3}
2	21000.0	3.600×10^{-1}	6.000×10^{-2}	2.700×10^{-1}
3	23000.0	4.500×10^{-2}	-1.000×10^{-4}	2.000×10^{-2}

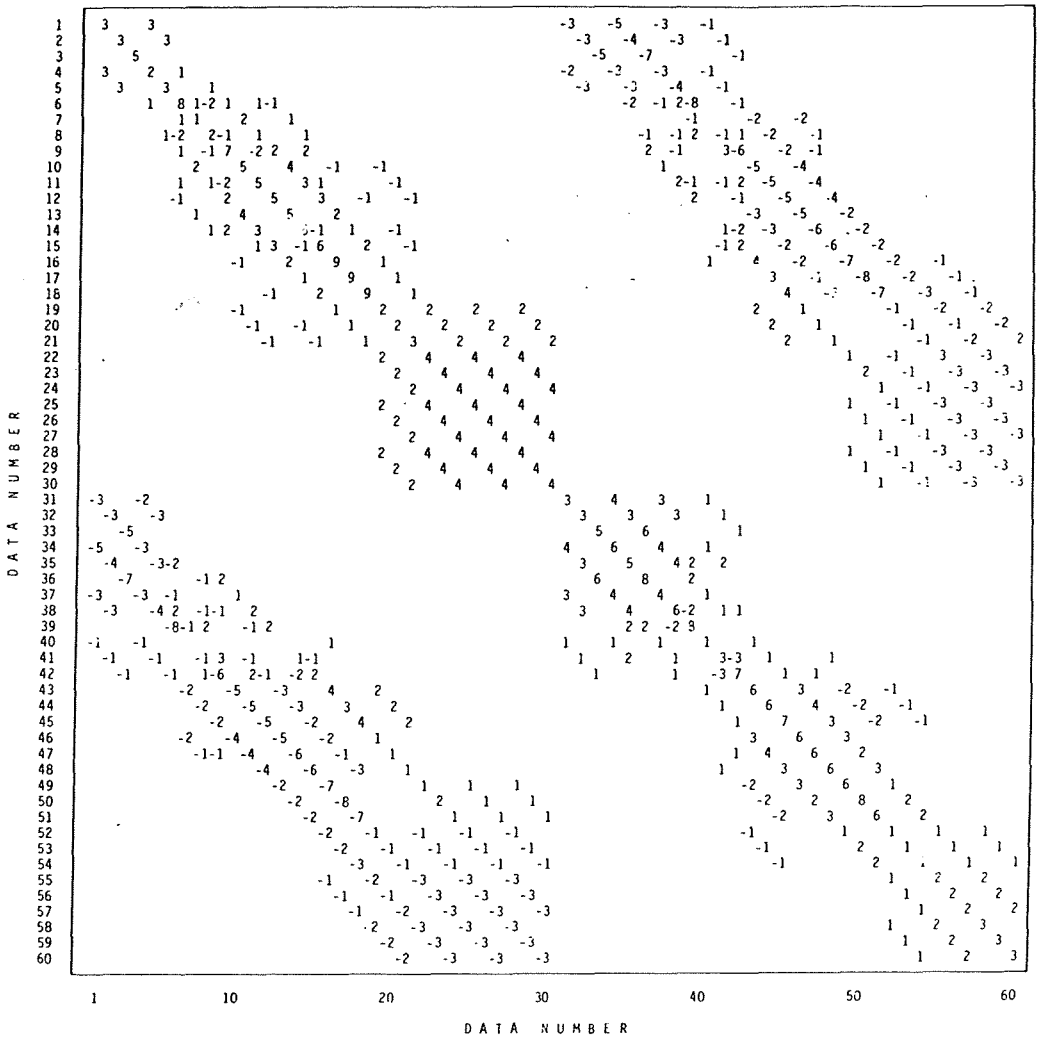

Figure 4. Information matrix for the model of Table 5, (Re, Im) data mode, $q = 14$.

When the second initial model is used the first cycle took eight iterations and the second cycle three, before the convergence criterion was satisfied. The final models for both cases coincide up to four significant digits in all of their parameters with model A as shown in Table 3.

In order to further test the inversion process a third initial model was tried under the same conditions as before. The parameters for this model are shown in Table 5. These parameters were obtained by perturbing the ones for the true model A, Table 3, by as much as

100 per cent. Even so, the generalized inversion converged again to a final model which coincides with model A up to four significant digits. This time the first cycle took 13 iterations to converge and the second cycle required three iterations.

The data used for the inversion were read by the program with five significant digits and hence contained a very small amount of noise. Consequently, we could afford to use a severe convergence criterion ($\epsilon = 10^{-5}$) and a rather liberal threshold criterion of 10^{-4} in the eigenvalue relative magnitude for the determination of q. Thus, all the iterations in example 1 were performed using the maximum number of effective degrees of freedom which is 14 in this case since one parameter is fixed. One should not expect to find such a nice situation when inverting noisier field data.

The resolution matrix for 14 degrees of freedom is the identity matrix, and the information matrix normalized to the open interval $(-10; 10)$ is shown in Fig. 4. It has several interesting features. The two 30×30 blocks on the main diagonal represent the real and the imaginary parts of the data whereas the off-diagonal blocks represent interdependencies between real and imaginary parts. This matrix shows clearly that each data tends to relate more closely to other data which represent the same impedance tensor elements rather than to data which represent other elements. This implies that the different impedance tensor elements contain relatively independent information. Hence we are led to an important conclusion. In order to invert tensorial magnetotelluric data an inversion procedure that makes use of the information included in all of the tensor elements is needed. A method which is based merely on any scalar mode, TE or TM, separately, might lack essential information.

For the same model we performed also a two-cycle iterative generalized inversion using the (ρ, ϕ) data mode (Table 1), under the same conditions as before and the initial model of Table 4. The convergence was slower, the first cycle required 21 iterations. Most of the iterations were done with $q = 13$. The final model is not as good as the one achieved using the (Re, Im) data mode. The parameters obtained are less accurate especially the half-space conductivity tensor elements which are in error of as much as 20 per cent. The nature of the resolution achieved in each of the parameters is revealed indeed by the model-evaluations means which are given by the generalized inversion.

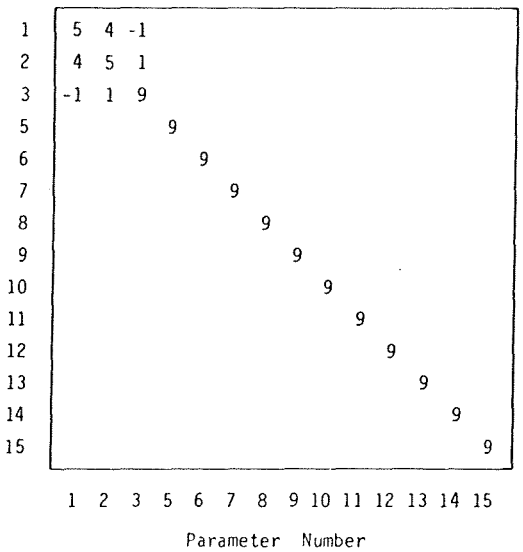

Figure 5. Resolution matrix for the model of Table 7, (ρ, ϕ) data mode, $q = 13$.

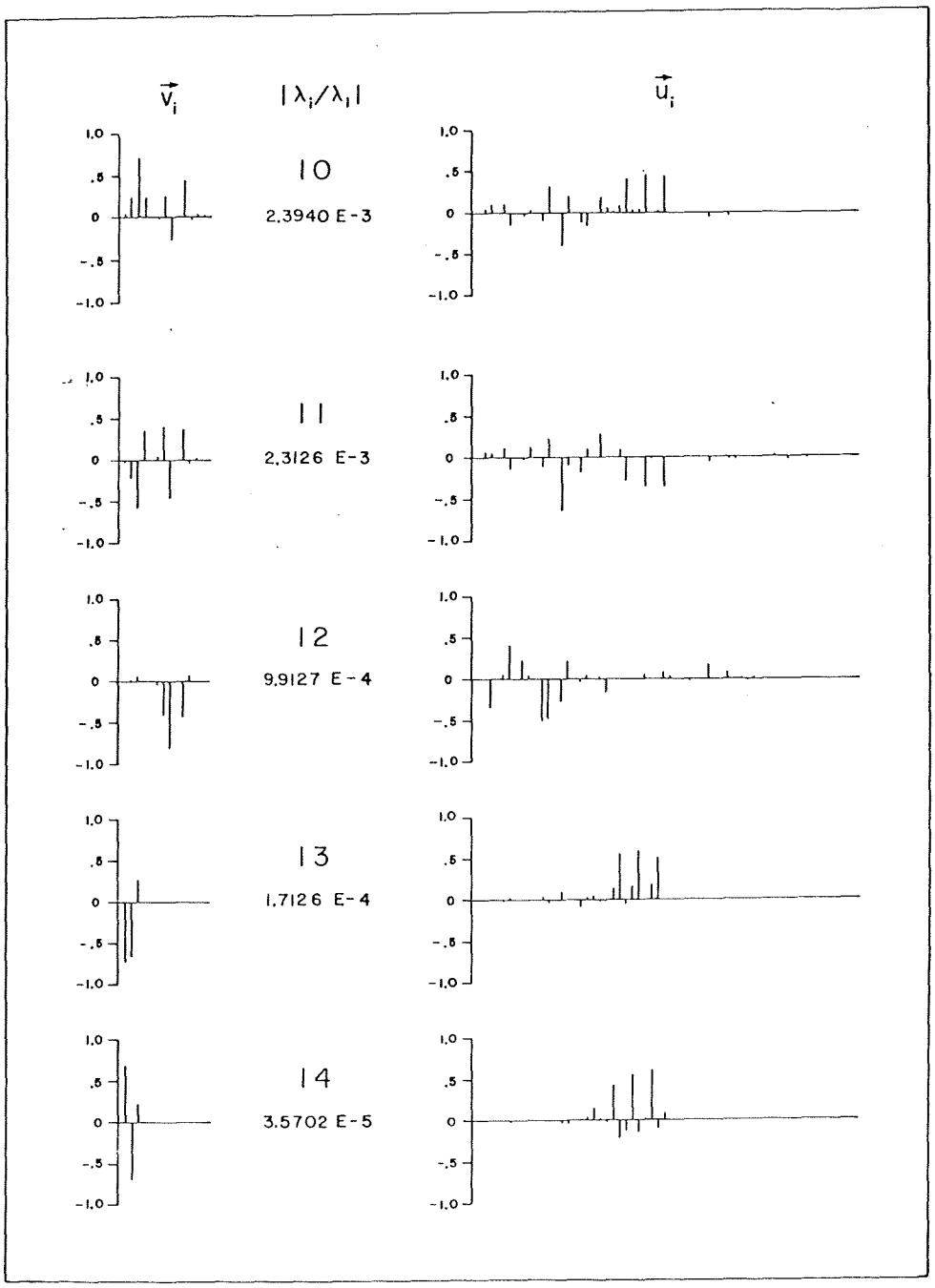

Figure 6. Last part of the eigensystem for the model of Table 7.

The resolution matrix for the final model and 13 degrees of freedom is shown in Figure 5. It is clearly seen that the conductivity parameters of the half-space are less resolved and more interdependent and consequently more affected by the equivalence problem. Another representation of the same quality is given by the double eigensystem in the forward resolution analysis for the final model, the last part of which is shown in Fig. 6.

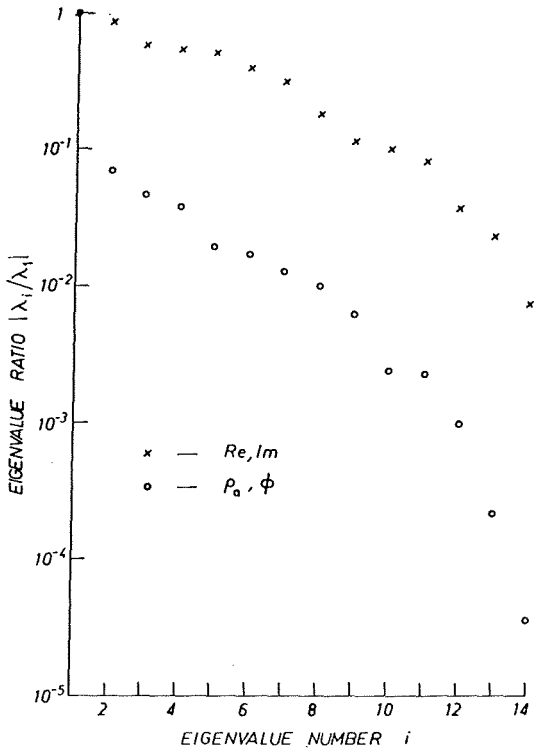

Figure 7. Eigenvalue ratio for two data modes.

A further comparison of the two data modes is demonstrated in Fig. 7 in which $|\lambda_i/\lambda_1|$ from the forward resolution analysis for both final models is shown. This eigenvalue ratio affects the inversion since it controls the data noise amplification as well as the effective number of degrees of freedom q determined using either an eigenvalue threshold or trade-off curves. It is clearly seen that we are much better off using the (Re, Im) data mode for this model. If the data were noisier we could use eigenvalue threshold which is 1.5 orders of magnitude higher than the 10^{-4} used here and still have $q = 14$ for the (Re, Im) data mode but only $q = 9$ for the (ρ, ϕ) data mode. It should be noted, however, that this conclusion holds true only for the final models, and the situation may be somewhat different for each of the intermediate models in the various iterative inversion steps.

We see that the (ρ, ϕ) data mode is not as strong in resolving model A parameters as the (Re, Im) one, which is in agreement with the observation made on the eigenvalues for another model (Abramovici *et al.* 1976). We do not claim, however, that the (Re, Im) data mode is better for any data and any model but only that different data modes differ in their resolving power with respect to various models, and therefore a forward resolution analysis should be performed for each case individually.

5.2 EXAMPLE 2

The use of the magnetotelluric anisotropic inversion is demonstrated here for a model which has one isotropic layer and two anisotropic layers. This model is similar to model A and was constructed by using the same parameters as in Table 2 for layers 2, 3 and for the half-space, and changing layer 1 into an isotropic one with conductivity of $2.560 \times 10^{-2} (\Omega m)^{-1}$. We used as an initial model for the inversion the same one as in Table 4, taking $\sigma_{12}^{(j)} = 0$,

$j = 0, \ldots, 3$ as if we do not know about the existence of an isotropic layer. The parameters of the final model obtained by the generalized inversion performed under the same conditions as before agreed with the exact model up to four significant digits.

All the iterations were performed using $q = 13$ since the smallest eigenvalue was practically zero. This is how the isotropic layer reveals its existence. Although the initial model had anisotropic conductivities for layer 1, the process came up with two values for the diagonal elements of the conductivity tensor which agreed up to four digits

$$\sigma_{11}^{(1)} = 2.5601 \times 10^{-2} \quad \text{and} \quad \sigma_{22}^{(1)} = 2.5599 \times 10^{-1}.$$

5.3 EXAMPLE 3

In the two previous examples the initial model had the same number of layers as the exact model. This number is not known generally. In this example the behaviour of the generalized inversion for a case in which the initial model has a 'wrong' number of layers is demonstrated. We consider again model A, for which the exact parameters in the principal system are shown in Table 3, and (Re, Im) data mode. The initial model used is given in Table 6. This model is actually the one of Table 4, with an extra layer which was added to it. It is a low-conductivity layer, which makes it even more difficult for magnetotelluric inversion.

The process which was performed with same conditions as in example 1 converged slower than in the other examples. The first cycle took 30 iterations and the second four before convergence was attained. The final model is shown in Table 7.

It is clearly seen that the process actually got rid of the extra layer by equalizing the conductivities on both sides of the interface at z_1 and thus making this interface insignificant. This is one way in which the generalized inversion is overcoming an initial model with too many layers. In some other cases the process may yield an insignificant layer by making it electrically very thin.

The generalized inversion conveyed the information about the final model through its evaluation means. Most of the iterations were done with $q = 17$. The resolution matrix normalized to the open interval $(-10; 10)$ for 17 degrees of freedom is diagonal. All of the diagonal elements are equal to 9 except the one corresponding to x_8 (i.e. z_1), which is equal zero classifying therefore this parameter is insignificant. A similar information is revealed by the eigensystem: x_8 is represented only in $\mathbf{v}_{18}$ which belongs to the smallest eigenvalue.

Table 6. Initial model for example 4.

Layer	z_i (m)	$\sigma_{11}^{(i)}$	$\sigma_{12}^{(i)}$	$\sigma_{22}^{(i)}$
HS	0.0	1.240	1.040×10^{-1}	2.600×10^{-1}
1	19000.0	1.900×10^{-2}	2.500×10^{-3}	1.300×10^{-2}
2	20000.0	1.000×10^{-3}	1.000×10^{-4}	2.000×10^{-3}
3	21500.0	2.300×10^{-1}	2.600×10^{-2}	1.800×10^{-1}
4	23000.0	7.500×10^{-2}	5.400×10^{-3}	3.000×10^{-2}

Table 7. Final model for example 4.

Layer	z_i (m)	$\sigma_{11}^{(i)}$	$\sigma_{12}^{(i)}$	$\sigma_{22}^{(i)}$
HS	0.0	9.956×10^{-1}	-5.872×10^{-2}	2.043×10^{-1}
1	3918.0	2.543×10^{-2}	1.671×10^{-3}	1.590×10^{-2}
2	20000.0	2.531×10^{-2}	1.670×10^{-3}	1.591×10^{-2}
3	22000.0	1.933×10^{-1}	2.547×10^{-2}	1.317×10^{-1}
4	23000.0	8.959×10^{-2}	-4.531×10^{-3}	4.041×10^{-2}

Conclusions

Based on the examples shown here we are certain that the presented generalized inverse procedure is able to produce geophysically significant anisotropic models, and to characterize their global significance. The iterative process involved is quite stable and the computer-time is not excessively high. It may be that in some cases it will be necessary to re-run the program several times after examining all the information in the eigenvectors, resolution matrix, etc, for every stage.

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