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A NUMERICAL INVESTIGATION OF THE THERMAL STATE OF THE EARTH'S MANTLE

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ABSTRACT

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An idealized convecting mantle with internal heat generation and viscosity dependent on temperature and pressure is examined with numerical calculations. Temperature and viscosity are coupled and self-regulating in the quasi-steady solutions. The lack of any tendency for upwelling flow to constrict itself to narrow channels argues against the existence of plumes. Unsteadiness is an essential feature of mantle convection, not only for mixing at large Rayleigh numbers, but also to prevent the flow from being impeded by continuous rigid regions.

Since the advent of the theory of plate tectonics, it has become more evident that the thermal state of the earth's interior must be influenced by mass flow in the mantle. Earlier theoretical work, taking into account heat transfer by conduction and radiation diffusion alone (MacDonald, 1959; Clark and Ringwood, 1964) is attractive for its relative simplicity and for its well-determined solutions. However, to attempt te be realistic, one must enter into the more ambiguous calculations of convection, with viscosity dependent on temperature and pressure. McKenzie et al. (1974) have thoroughly investigated models with constant viscosity, and Houston and DeBremaecker (in press) have calculated models with temperature-dependent viscosity. In both these investigations it was assumed that the seismic discontinuity at 700 km depth is a barrier to convection.

Radioactive heat generation and a functional dependence of viscosity on temperature and pressure are prescribed as part of the input to these calculations. Quasi-steady temperature and viscosity fields are obtained as output. Such models can exhibit the feature found by Tozer (1970) that temperature and viscosity are coupled together and assume values that allow the transport out of heat at the rate that it is generated. The question of whether there is a limit on the depth of convective flow is determined in the same self-regulating manner by the pressure-dependence of viscosity.

If a substantial part of the heat flowing out of the earth's surface originates not in the crust but throughout the mantle or core, then the mantle is highly unstable. If one accepts a thermal conductivity not sensitive to temperature (Schatz and Simmons, 1972) and a uniform viscosity of 10^{22} poise determined from glacial rebound (O'Connell, 1971) then the Rayleigh number for an internally heated system (Roberts, 1967) for the mantle as a whole is greater than 10^8. With MacDonald's estimate of thermal conductivity it is reduced one order of magnitude. With a Rayleigh number this large one cannot expect convection to be steady. The unsteadiness will not appear in the form of turbulence, but rather as shifting patterns of large-scale eddies. Lithospheric plates are part of the overall circulation. Norman Sleep (personal communication, 1970) has emphasized that the nonsteady pattern of surface plates, with changing distances from ridges to trenches, obviously implies nonsteady circulation within the mantle.

The numerical calculations are time-dependent. We seek to approach quasi-steady solutions, in which horizontal averages of temperature are steady. In this search for a self-regulating mantle some idealizations are made. A uniform value of radioactive heat generation of 10^{-14} cal. cm^{-3} sec^{-1} is assumed throughout the mantle. This rate of total heat generation equals the total heat flow at the earth's surface. The calculations are two-dimensional. If a spherical sector of the mantle, 2900 km deep, is flattened, keeping the same volume beneath a unit area of surface, the depth becomes 1800 km. The calculational region is chosen to be a square 1800 km on a side. Heat flow at the bottom boundary is zero. Phase changes are ignored, and uniform composition is assumed.

In a steady state, surface heat flow must equal the internal heat generation. By the principle of conservation of energy, heat from viscous dissipation cannot appear in the overall energy balance. Viscous heating, which might be important locally, is compensated by adiabatic temperature changes throughout the convecting volume. (See the Appendix.) In this work both viscous heating and adiabatic temperature changes are ignored. Adiabatic temperature changes with depth are approximated by superimposing on the temperature solution a gradient of 0.43 deg/km. In other respects the equations used were the same as in Andrews (1972).

The first calculation was naively attempted with a small constant value of thermal conductivity. The solution was highly unsteady, and would have required an inordinate amount of computer time to approach a quasi-steady state. For that reason the specification of conductivity was changed to be:

$$K = a + bT^3$$

where T is absolute temperature and the coefficients are:

$a = 6 \cdot 10^{-3}$ cal.cm^{-1}sec^{-1}deg^{-1}

$b = 5 \cdot 10^{-12}$ cal.cm^{-1}sec^{-1}deg^{-4}

This function closely approximates MacDonald's function.

Viscosity is specified as:

$$\eta = \eta_0 \exp\,[(E^* + PV^*)/RT]$$

when η_0 is chosen to give $\eta = 3 \cdot 10^{21}$ poise at 200 km depth if $T = 1673\ ^\circ$K (1400 °C). The choice of η_0 was the most arbitrary of this work. Recently published estimates of the effective viscosity of olivine at 1400 °C, 70 kbar, and shear stress of 10 bars range from $3 \cdot 10^{23}$ poise (Stocker and Ashby, 1973) to 10^{20} poise (Kirby and Raleigh, 1973). The activation energy adopted:

$$E^* = 100 \text{ kcal/g-atom}$$

is in the mid-range of published estimates (Gordon, 1965; Goetze, 1971; Weertman, 1970; Stocker and Ashby, 1973). Two different values of activation volume are considered, $V^* = 5$ cm^3/g-atom and $V^* = 6$ cm^3/g-atom, both near the lower end of the range of published estimates. To avoid numerical difficulties an upper limit on viscosity of $5 \cdot 10^{24}$ poise was imposed.

Other physical parameters are:

density	$\rho_0 = 3.4$ g/cm^3
acceleration of gravity	$g = 990$ cm/sec^2
thermal expansivity	$\alpha = 3 \cdot 10^{-5}$ deg^{-1}
heat capacity	$c_p = 0.311$ cal.g^{-1} deg^{-1}

The lithosphere is included in the calculation, for the variable viscosity allows it to be considered as part of the single fluid. However, it is important to account for deformation of the lithosphere by earthquakes in an approximate manner. A change of strain of $3 \cdot 10^{-4}$ in a major earthquake with a repeat time of 100 years gives an average strain rate of 10^{-13} sec^{-1}. This strain rate occurs only if stress is large enough to cause earthquakes — let us say 1 kilobar. Then the effective viscosity is 10^{22} poise, much smaller than the actual viscosity of a cool plate, and it can have a significant effect on the flow. The effective viscosity is highly nonlinear, for it does not allow shear stress to rise above 1 kilobar. Therefore, a yield stress of 1 kilobar is imposed in the numerical calculation.

The temperature field at the start consisted of a small horizontal temperature gradient designed to start convection superimposed on an adiabatic temperature field. An adiabatic temperature gradient is unstable in the case of internal heating.

In order to reach a quasi-steady state, the calculation was run for about $3 \cdot 10^9$ years. The calculation was done in stages of about 100 million years, and the output was examined after each stage, before restarting the computer calculation. Some qualitative observations of the unsteady solution may be informative.

The magnitude of the fluid velocity never exceeded 10 cm/year. After an initial transient, there was no tendency for upwelling to be concentrated in a narrow plume, and no tendency for any upwelling to remain in the same location. These observations argue against the plume hypothesis (Morgan, 1972) due to thermal effects alone. A chemical plume is possible, but it cannot be expected to remain in the same location.

After conductivity was increased midway in the calculation, the solution became smoother, and temperature in the lower mantle decreased slowly, changing 200 degrees in 10^9 years. Fluid velocities in the upper mantle changed little with the change in conductivity, remaining in the range 1/2—2 cm/year. This confirms the conclusion of Tozer (1965) that velocity is insensitive to conductivity.

The unsteadiness of the convective flow never manifested itself as turbulence or small-scale eddies, but rather as a shifting of the pattern of large-scale eddies at a velocity comparable to the fluid velocity.

The one stable element of the flow was the down-going lithosphere, which never left the boundary of the calculational region. Although it warms up somewhat during its descent, it cools neighboring material sufficiently to form a rigid pillar reaching into the lower mantle. Both side boundaries are planes of reflection for both temperature and stream function, for this is the only reasonable prescription to use if an overall energy balance is to be obtained. Unfortunately, this means that the down-going slab has a double thickness and warms up only a quarter as rapidly as it should. After the down-going slab was established, a perturbation of the solution could not change its location.

Calculations were run until horizontal averages of temperature were steady. After such a quasi-steady state was reached for the case $V^* = 5$ cm^3/g-atom, a calculation was started with $V^* = 6$ cm^3/g-atom, starting from that solution. A period of 1200 m.y. was required to reach a new quasi-steady state.

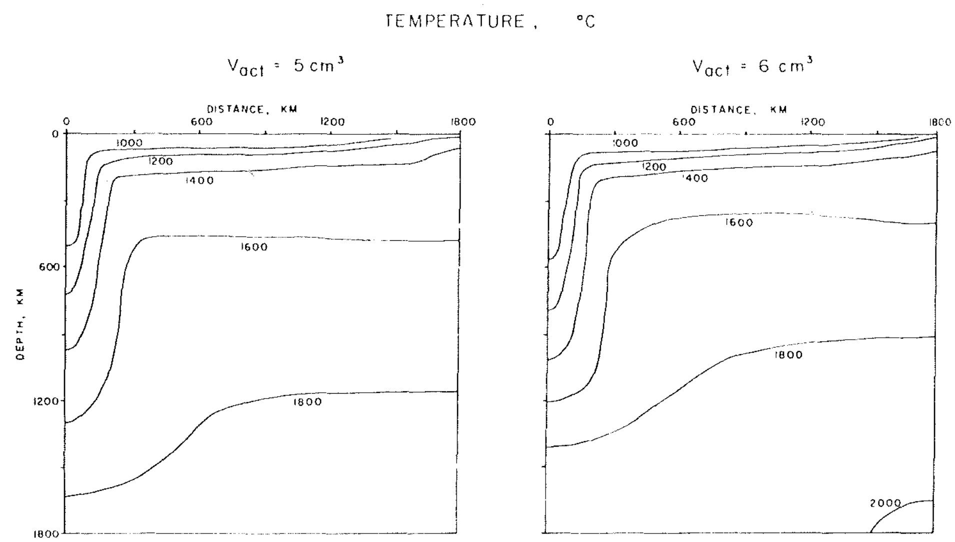

Fig. 1. Isotherms are shown for quasi-steady solutions with two different values of activation volume. Both the right and left boundaries are planes of reflection.

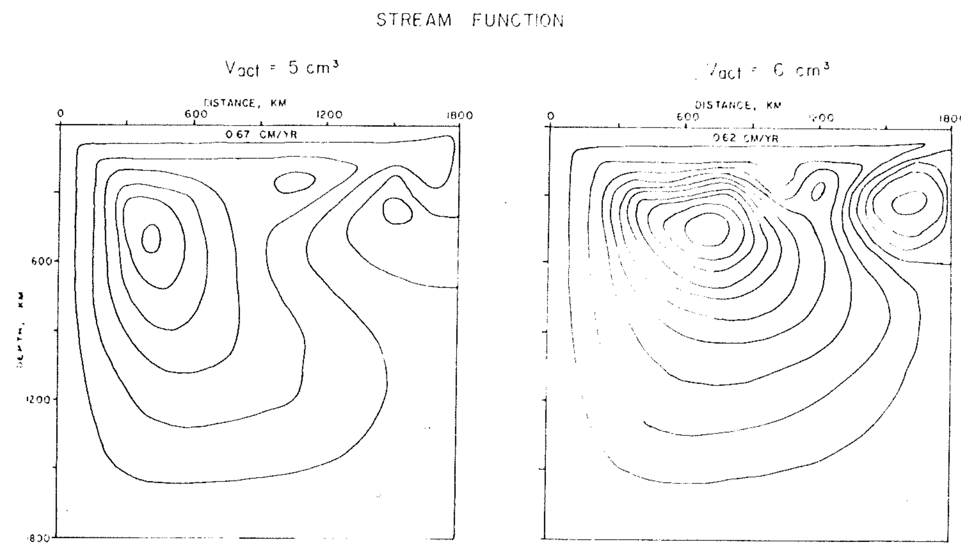

Fig. 2. Streamlines for the two cases shown in Fig. 1. The counter eddies in the upwelling region on the right in each case are not steady features.

Quasi-steady temperature contours for the two cases are shown in Fig. 1. The cold down-going slab is seen on the left side in each case, and new material is being added to the surface plate on the right. Temperature in the lower mantle is higher for the case $V^* = 6$ cm^3/g-atom, and there is a larger temperature gradient in the upper mantle.

Stream function for the two cases is shown in Fig. 2, using the same contour interval between streamlines in each case. Velocity in the upper mantle is larger in the case $V^* = 6$ cm^3/g-atom because of the larger temperature gradient. Flow in the upper mantle is not steady. In both solutions shown there is a counter eddy under the ridge. This counter eddy shifts position as time proceeds, so that flow under the ridge does not remain in the same direction.

The velocity of the surface plate and down-going slab is smaller in the case $V^* = 6$ cm^3/g-atom, despite larger velocity in the asthenosphere. The explanation may be found from the viscosity contours plotted in Fig. 3. The material of the down-going slab and its surroundings is cool and has a high viscosity. It extends down to the lower mantle and joins high-viscosity material there. The down-going slab is impeded by this continuous cool rigid region. With a larger activation volume, viscosity is larger in the lower mantle, and the velocity of the surface plate is smaller.

Regions in which the yield stress is reached are stippled in Fig. 3. Yielding occurs where the plate bends at the subduction zone, as expected. Yielding also occurs in the lower mantle in the down-going slab and the cool material around it. The slab is constrained by the boundary of the calculational

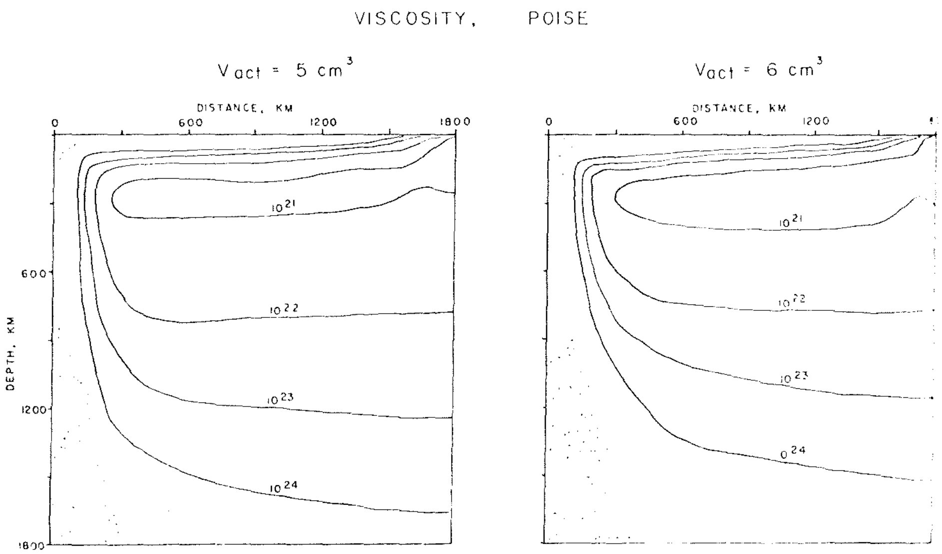

Fig. 3. Viscosity contours for the two cases shown in Fig. 1. The stippled areas are at the yield stress of 1 kilobar.

region to continue going down in the same place at all times. Therefore a cool rigid pillar standing on the bottom boundary is formed.

Temperature averaged horizontally is plotted as a function of depth in Fig. 4. The dashed line in the figure shows the adiabatic temperature gradient. The temperature gradient found in both cases is subadiabatic, as is to be expected for an internally heated fluid (McKenzie et al., 1974). In Fig. 4 the horizontal harmonic mean of viscosity is also plotted as a function of depth. The curve labelled 5 is for the case $V^* = 5$ cm^3/g-atom. When the activation volume was changed to $V^* = 6$ cm^3/g-atom, viscosity jumped to the dashed curve. The curve labelled 6 is the horizontal harmonic mean of viscosity after a new quasi-steady temperature field was established. The self-regulating feature of a convecting system with a variable viscosity is evident here. Limitations of computer time prevented examination of larger values of V^*, for which there might be a natural lower limit to convection.

It is clear from these solutions that subduction that remains at one location is not the most favored mode of flow. If the subduction zone could migrate in the calculation, then old subducted material would not impede continuing subduction, and larger plate velocities would result. Since the flow is confined in a box in this calculation, the subduction zone does not migrate. The horizontal average of the temperature field found under this constraint is an upper limit on results expected from less constrained cases, in which flow velocities will be larger.

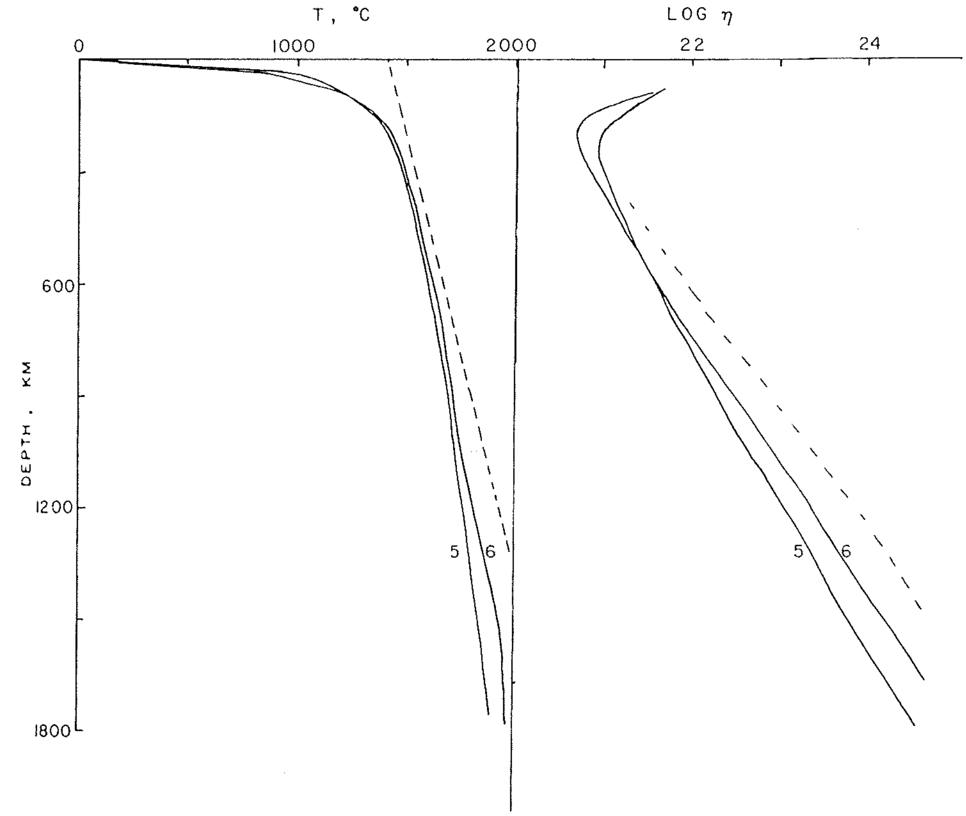

Fig. 4. Left. Horizontal average of temperature as a function of depth for activation volume equal to 5 cm^3/g-atom and 6 cm^3/g-atom. The dashed line shows the adiabatic gradient.
Right. Horizontal harmonic mean of viscosity for the two quasi-steady solutions (solid curves). Viscosity calculated with $V^* = 6$ cm^3/g-atom with the temperature solution for $V^* = 5$ cm^3/g-atom is shown as a dashed line to indicate sensitivity to this parameter and the self-regulating feature of the quasi-steady solutions.

CONCLUSIONS

This work provides an example of the approach required for a theoretical investigation of the thermal state of the earth's mantle.

The mantle is unstable enough that it is unlikely that a compositional stratification has been established. Therefore a depth limit for convection will be determined by pressure-dependence of viscosity. The entire depth of the mantle will be a self-regulating system.

Viscosity increases with depth in the quasi-steady solutions, and would increase even more rapidly with a larger activation volume, contrary to conclusions drawn from glacial rebound data.

Unsteadiness is an essential feature of convective flow in the mantle. It is required, not only by the large Rayleigh number of the warmer part of the flow, but also as a mechanism to mix and reduce the rigidity of the cooler portion of the mantle.

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APPENDIX

Viscous dissipation in a convecting system

For any velocity field in a viscous fluid the rate of heat generation per unit volume due to viscous dissipation is positive definite, and its integral over the fluid volume is nonzero. The principle of conservation of energy in a naturally convecting system in steady state requires that heat flowing out of the fluid equals heat flowing into the fluid plus radioactive heat generated within. Viscous heating cannot contribute to the overall energy balance. The paradox is resolved if the convecting fluid is recognized as a heat engine doing work upon itself (Jeffreys, 1930, 1956). Then the viscous dissipation integrated over the volume of the fluid must equal the work of thermal expansion integrated over the fluid, and this work must be derived from part of the heat flowing through the fluid. This conclusion, which must hold in general, will be illustrated in the Boussinesq approximation.

In the Boussinesq approximation the velocity field is required to be incompressible, but density changes due to thermal expansion are taken into account in the force balance. The Boussinesq approximation does not conserve mass locally, so it is not surprising that paradoxes arise in regard to conservation of energy. Density is taken to be:

$$\rho = \rho_0 - \rho_0 \alpha T$$

Hydrostatic pressure due to the constant reference density:

$$P = \rho_0 \, g \, y$$

is subtracted from the stress tensor, leaving stress components of a smaller order of magnitude.

In the general case let u_i be the velocity vector of the fluid and σ_{ij} be the stress tensor arising from an arbitrary rheology. Summation is implied over repeated subscripts, and a subscript following a comma indicates differentiation with respect to that coordinate. The rate of change of internal energy of a volume element due to mechanical work is stress times strain rate, and its volume integral can be related to other work integrals as follows:

$$\int u_{i,j}\sigma_{ij} \, \mathrm{d}V = \int (u_i \, \sigma_{ij})_{,j} \, \mathrm{d}V - \int u_i \, \sigma_{ij,j} \, \mathrm{d}V$$

$$= \int u_i \, \sigma_{ij} \, \mathrm{d}S_j \; + \int \rho \, g_i u_i \, \mathrm{d}V \qquad (1)$$

The first term on the right-hand side of the equation is work done at the boundaries, and is assumed to be zero. Inertial force is negligible for flow in the earth, so the force density from the stress tensor is balanced by the gravitational body force:

$$\sigma_{ij,j} = -\rho \, g_i$$

Then the second term on the right is the rate of change of gravitational potential energy, and is zero for a steady closed circulation. Therefore the left-hand side of the equation, the internal work done in the fluid, is zero. The shear contribution to the integral (which is positive for a viscous fluid) must be cancelled by the dilatational contribution.

The Boussinesq approximation violates these general considerations, and some careful consideration is needed to conserve energy. For an incompressible fluid the left-hand side of eq. 1 is the total viscous dissipation, W_D, and is nonzero. The second term on the right-hand side of eq. 1 is also nonzero. It is a fictitious net work done by the buoyant force:

$$W_\mathrm{B} = -\rho_0 \, \alpha g_i \int T \, u_i \, \mathrm{d}V$$

and is positive, since temperature is larger where velocity is opposite to the direction of gravity. An integration by parts, as in eq. 1, yields the result:

$$W_\mathrm{D} = W_\mathrm{B}$$

in the Boussinesq approximation.

While maintaining the approximation of incompressibility in the determination of the velocity field, we must not neglect the product of a small density change times the large hydrostatic pressure in the energy balance. The work due to adiabatic density change is:

$$W_{\mathrm{A}} = \int P\, u_{i,i}\, \mathrm{d}V$$

In the Boussinesq approximation:

$$u_{i,i} = \alpha u_i\, T_{,i} + \alpha \frac{\partial T}{\partial t}$$

The integral of the time derivative of temperature will drop out if horizontal averages of temperature are steady. Then:

$$W_{\mathrm{A}} = \rho_0\, g\, \alpha \int y\, u_i\, T_{,i}\, \mathrm{d}V$$

After making the approximation:

$$u_i\, T_{,i} \cong (u_i\, T)_{,i}$$

one can integrate by parts and find:

$$W_{\mathrm{A}} = W_{\mathrm{B}} = W_{\mathrm{D}}$$

Therefore energy can be conserved in the Boussinesq approximation if adiabatic work is taken into account in the determination of temperature. A complete thermodynamic description of the heat engine requires another thermal variable, such as entropy or a thermodynamic potential. Elimination of all thermal variables except temperature yields terms representing adiabatic temperature changes with depth and also adiabatic temperature changes at constant depth as the fluid gains or loses heat (McKenzie, 1968). The latter term, integrated over the volume, must be equal and opposite to the integrated temperature increase from viscous heating. The differential equation is developed by Turcotte et al. (1974).

The relative importance of all these work terms is measured by the dimensionless number:

$$D = \frac{\alpha\, g\, h}{c_p}$$

where h is the depth of the fluid (Turcotte et al., 1974). For a depth of 2000 km in the earth's mantle, $D \cong 0.5$. Using the definition of Grüneisen's parameter Γ, an equivalent expression can be found:

$$D = \Gamma \frac{\rho\, g\, h}{k_s}$$

where k_s is the isentropic bulk modulus. Grüneisen's parameter for any material is of order one. Therefore, D, which has been called the dissipation number, is described more graphically as the compression ratio of the heat engine.