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A STUDY OF PARTIAL PENETRATION IN A TWO-LAYERED AQUIFER

I. ANALYTICAL SOLUTION

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ABSTRACT

The behavior of a layered aquifer under the influence of a pumping well is a problem of interest in the fields of hydrogeology, geothermal engineering, and petroleum engineering. Numerous papers have been written on various aspects of this problem. Hantush and Jacob (1955) have presented solutions for steady state flow to a well draining one of the layers of a two-layered bounded aquifer. Lefkovits et al. (1961) studied the transient performance of a stratified bounded reservoir where the producing well is completely penetrating and there is no crossflow. Papadopulos (1966) has studied the same problem for only two layers of infinite areal extent. A similar problem, but with crossflow between adjacent layers, has also been investigated by Katz (1960) and Russell and Prats (1962) for the case of constant head at the wellbore, and by Jacquard (1960) for constant flow rate.

In addition to the above works, which are all based on the analytical approach, many authors have applied numerical as well as analog models to handle problems of flow in layered aquifers (Vacher and Cazbat, 1961; Pizzi et al., 1965; Javandel and Witherspoon, 1968, 1969; Neuman and Witherspoon, 1969; Kazemi and Seth, 1969). Recently, Javandel and Witherspoon (1979) studied the problem of flow to a partially penetrating well in a two-layered aquifer where the well is open in the top layer and the lower layer is considered to be infinitely thick.

In this paper we shall present an analytic solution to the problem of transient flow to a partially penetrating well that is open in either layer of finite thickness in a two-layered system. Crossflow is permitted at the interface between the two layers. Closed form solutions have been obtained which can easily be evaluated numerically. Simplified forms of the solutions for small and large values of time have been developed from the main solution. It has also been shown that the solution reduces to the case of single layer partial penetration once we allow the permeability of the nonperforated layer to vanish. The approach here is to start with the problem when the pumping well is open only in the top layer. A second solution is also developed when the well is partially penetrating only in the lower layer. A numerical evaluation of these solutions and the application of the results to the interpretation of field problems will be presented in a subsequent paper.

INTRODUCTION

The behavior of a layered aquifer under the influence of a pumping well is a problem of interest in the fields of hydrogeology, geothermal engineering, and petroleum engineering. Numerous papers have been written on various aspects of this problem. Hantush and Jacob (1955) have presented solutions for steady state flow to a well draining one of the layers of a two-layered bounded aquifer. Lefkovits et al. (1961) studied the transient performance of a stratified bounded reservoir where the producing well is completely penetrating and there is no crossflow. Papadopulos (1966) has studied the same problem for only two layers of infinite areal extent. A similar problem, but with crossflow between adjacent layers, has also been investigated by Katz (1960) and Russell and Prats (1962) for the case of constant head at the wellbore, and by Jacquard (1960) for constant flow rate.

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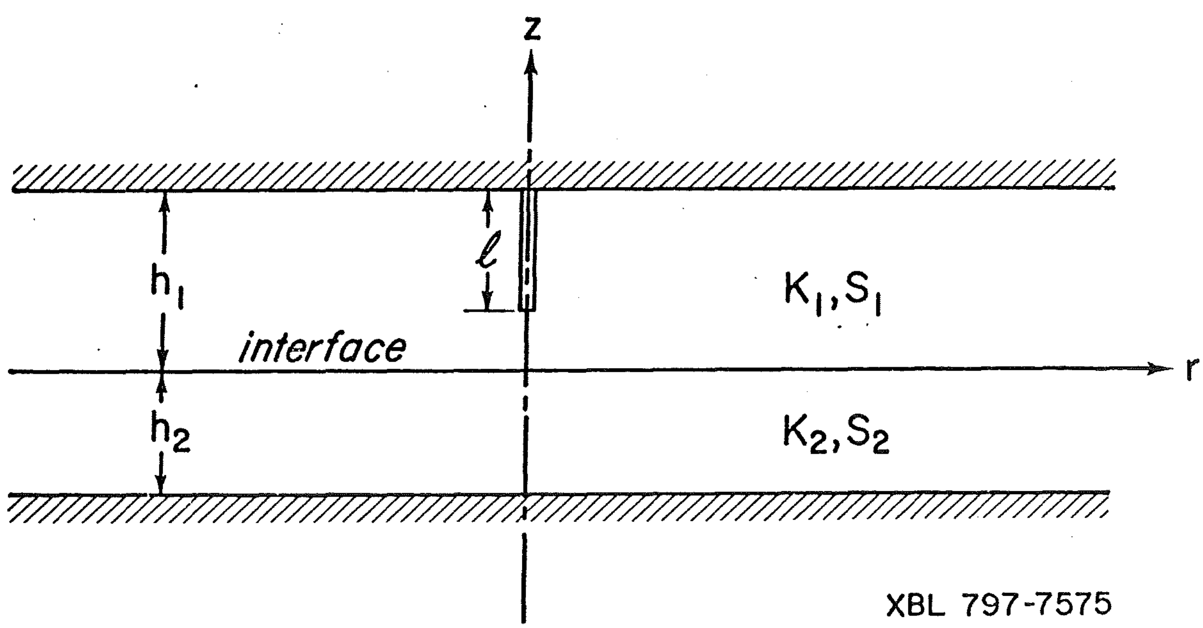

Figure 1. Schematic diagram of a two-layered aquifer with a partially penetrating well in the upper layer.

WELL OPEN IN THE TOP LAYER

Let us consider an aquifer consisting of two layers that is contained above and below by impervious layers as illustrated on Figure 1. Each layer has its own flow properties, is finite in thickness, and extends radially to infinity. The interface between the two layers is an open boundary, meaning that no discontinuity of potential or its gradient is allowed across this surface. The top layer of the system is partially penetrated by a well of infinitesimal radius for a length ℓ from the top of the aquifer. If the well is pumped at a constant rate, Q, we are interested in determining the value of drawdown, s(r, z, t), at any point after pumping starts. The differential equations and initial and boundary conditions to describe this problem can be written as: *

$$\frac{\partial^2 s_i}{\partial r^2} + \frac{1}{r}\frac{\partial s_i}{\partial r} + \frac{\partial^2 s_i}{\partial z^2} = \frac{1}{\alpha_i}\frac{\partial s_i}{\partial t} \qquad i = 1, 2 \tag{1}$$

$$s_i(r, z, 0) = 0 \tag{2}$$

$$\frac{\partial s_1}{\partial z}(r, h_1, t) = 0 \tag{3}$$

$$\frac{\partial s_2}{\partial z}(r, -h_2, t) = 0 \tag{4}$$

$$\lim_{r\to\infty} s_i(r, z, t) = 0 \tag{5}$$

$$s_1(r, o, t) = s_2(r, o, t) \tag{6}$$

$$K_1 \frac{\partial s_1}{\partial z}(r, o, t) = K_2 \frac{\partial s_2}{\partial z}(r, o, t) \tag{7}$$

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* Note: Explanation of all terms is given in the Notation.

$$\lim_{r \to 0} \left(r \frac{\partial s_1}{\partial r} \right) = - \frac{Q}{2\pi K_1 \ell} \qquad \text{for } (h_1 - \ell) < z < h_1 \qquad (8)$$

$$\lim_{r \to 0} \left(r \frac{\partial s_i}{\partial r} \right) = 0 \qquad \text{for } -h_2 < z < (h_1 - \ell) \qquad (9)$$

In order to handle the nonuniform boundary condition along the axis of the well, one can arbitrarily divide the top layer of the aquifer into two separate layers by considering an imaginary interface at the elevation of $z = h_1 - \ell$. The system is then made of three layers, two of them having the same flow properties. Let us then designate three different symbols for drawdown: s_1 for the top layer in the zone between the top of the aquifer and an imaginary plane passing through the elevation at the bottom of the well; s_2 for the bottom layer; and s_3 for the zone between the elevation at the bottom of the well and the top of the lower layer.

The solution of the problem can be obtained by successive application of Laplace and Hankel transformations over t and r, respectively. If we indicate the Laplace transform of $s_i(r, t)$ by $\bar{s}_i(r, p)$ and the Hankel transform of $\bar{s}_i(r, p)$ by $\bar{\bar{s}}_i(\xi, p)$, then equations 1 through 9 become:

$$\frac{d^2\bar{\bar{s}}_1}{dz^2} - \omega_1^2 \bar{\bar{s}}_1 = \frac{-Q}{2\pi \ell K_1 p} \qquad \text{for } (h_1 - \ell) < z < h_1 \qquad (10)$$

$$\frac{d^2\bar{\bar{s}}_2}{dz^2} - \omega_2^2 \bar{\bar{s}}_2 = 0 \qquad \text{for } -h_2 < z < 0 \qquad (11)$$

$$\frac{d^2\bar{\bar{s}}_3}{dz^2} - \omega_1^2 \bar{\bar{s}}_3 = 0 \qquad \text{for } 0 < z < (h_1 - \ell) \qquad (12)$$

$$\frac{d\bar{\bar{s}}_1}{dz}(\xi,\ h_1,\ p) = 0 \tag{13}$$

$$\frac{d\bar{\bar{s}}_2}{dz}(\xi,\ -h_2,\ p) = 0 \tag{14}$$

$$\bar{\bar{s}}_1(\xi,\ h_1 - \ell,\ p) = \bar{\bar{s}}_3(\xi,\ h_1 - \ell,\ p) \tag{15}$$

$$\bar{\bar{s}}_3(\xi,\ 0,\ p) = \bar{\bar{s}}_2(\xi,\ 0,\ p) \tag{16}$$

$$\frac{d\bar{\bar{s}}_1}{dz}(\xi,\ h_1 - \ell,\ p) = \frac{d\bar{\bar{s}}_3}{dz}(\xi,\ h_1 - \ell,\ p) \tag{17}$$

$$K_2\ \frac{d\bar{\bar{s}}_2}{dz}(\xi,\ 0,\ p) = K_1\ \frac{d\bar{\bar{s}}_3}{dz}(\xi,\ 0,\ p) \tag{18}$$

where $\omega_1 = \left(\frac{p}{\alpha_1} + \xi^2\right)^{\frac{1}{2}}$ and $\omega_2 = \left(\frac{p}{\alpha_2} + \xi^2\right)^{\frac{1}{2}}$

Equations 10 through 12 are now simple, ordinary differential equations whose solutions may be readily written as:

$$\bar{\bar{s}}_1 = C_1\cosh[\omega_1(z - h_1)] + \frac{Q}{2\pi\ell K_1 p\omega_1^2} \tag{19}$$

$$\bar{\bar{s}}_2 = C_2 \cosh\ [\omega_2(z + h_2)] \tag{20}$$

$$\bar{\bar{s}}_3 = A\ \sinh(\omega_1 z) + B\ \cosh(\omega_1 z) \tag{21}$$

Note that conditions 13 and 14 have already been considered in writing equations 19 and 20.

Constants A, B, C_1, and C_2 can be found through application of boundary conditions 15 through 18. Substituting the expressions for the above constants in equations 19 through 21 and performing the Hankel transform inversion, one can obtain:

$$\bar{s}_1 = \frac{Q}{2\pi \ell K_1} \int_0^\infty \left\{ \frac{1}{p\omega_1^2} - \frac{\cosh[\omega_1(z - h_1)]}{p\omega_1^2} \cdot \frac{K_2\omega_2 \sinh(\omega_2 h_2)\cosh[\omega_1(h_1 - \ell)] + K_1\omega_1\cosh(\omega_2 h_2)\sinh[\omega_1(h_1 - \ell)]}{FF(\omega_1, \omega_2)} \right\} \cdot J_0(\xi r)\xi \, d\xi \qquad (22)$$

$$\bar{s}_2 = \frac{Q}{2\pi \ell K_1} \int_0^\infty \frac{\cosh[\omega_2(z + h_2)]}{p\omega_1^2} \cdot \frac{K_1\omega_1 \sinh(\omega_1 \ell) \cdot J_0(\xi r)\, \xi \, d\xi}{FF(\omega_1, \omega_2)} \qquad (23)$$

$$\bar{s}_3 = \frac{Q}{2\pi \ell K_1} \int_0^\infty \frac{\sinh(\omega_1 \ell)}{p\omega_1^2} \cdot \frac{K_2\omega_2 \sinh(\omega_2 h_2)\sinh(\omega_1 z) + K_1\omega_1\cosh(\omega_2 h_2)\cosh(\omega_1 z)}{FF(\omega_1, \omega_2)} \cdot J_0(\xi r)\xi \, d\xi \qquad (24)$$

where

$$FF(\omega_1, \omega_2) \equiv K_2\omega_2\sinh(\omega_2 h_2)\cosh(\omega_1 h_1) + K_1\omega_1\cosh(\omega_2 h_2)\sinh(\omega_1 h_1).$$

Equations 22 through 24 represent the Laplace transform solutions for drawdowns in the aquifer.

To obtain the inverse solutions of equations 22 through 24, let us first consider

$$\bar{G}(p) \equiv \frac{P(p)}{g(p)} \equiv \frac{\cosh[\omega_1(z - h_1)]}{p\omega_1^2} \tag{25}$$

$$\cdot \frac{K_2\omega_2\sinh(\omega_2h_2)\cosh[\omega_1(h_1 - \ell)] + K_1\omega_1\cosh(\omega_2h_2)\sinh[\omega_1(h_1 - \ell)]}{FF(\omega_1, \omega_2)}$$

If the zeros of g(p) are shown by $p_1, p_2, p_3, \ldots, p_n, \ldots$ such that each of them has a different value, provided that $P(p_n) \neq 0$ and $g'(p_n) \neq 0$, then the inverse transform of $\bar{G}(p)$ may be obtained from the following formula, Jaeger (1949):

$$G(t) = L^{-1}\{\bar{G}(p)\} = \sum_{n=1}^{\infty} \frac{P(p_n)}{g'(p_n)} e^{p_nt} \tag{26}$$

Any of the summation terms in equation 26 may be replaced by

$$\left[\frac{(p - p_n)P(p)}{g(p)}\right]_{p=p_n} e^{p_nt}.$$

The zeros of g(p), as defined in equation 25 are $p = 0$, $p = -\xi^2\alpha_1$, (equivalent to $\omega_1^2 = 0$), as well as all zeros of

$$FF(\omega_1, \omega_2) = K_2\omega_2\sinh(\omega_2h_2)\cosh(\omega_1h_1) + K_1\omega_1\cosh(\omega_2h_2)\sinh(\omega_1h_1) = 0 \tag{27}$$

Depending on the nature of ω_1 and ω_2, four different cases should be considered.

<u>Case 1</u>. When both ω_1 and ω_2 are real. The left hand side of equation 27 is always greater than zero and, as a result, the equation has no zeros for such a case.

Case 2. If both ω_1 and ω_2 are purely imaginary, then we may introduce the following change of variable,

$$\omega_1 = \pm i\beta/h_1 \qquad \text{and} \qquad \omega_2 = \pm i\gamma/h_2,$$

where β and γ are both real and positive. Equation 27 may now be written as

$$A\gamma_n \tan\gamma_n + \beta_n \tan\beta_n = 0 \tag{28}$$

where

$$A = \frac{K_2 h_1}{K_1 h_2} \qquad \text{and} \qquad \beta_n = \sqrt{h_1^2\left[\frac{\alpha_2}{\alpha_1}\left(\frac{\gamma_n^2}{h_2^2} + \xi^2\right) - \xi^2\right]}$$

Equation 28 has an infinite number of zeros such as $\gamma_1, \gamma_2, \gamma_3, \ldots \gamma_n, \ldots$, and the corresponding values of p_n are given by:

$$p_n = -\alpha_2\left(\frac{\gamma^2{}_n}{h_2^2} + \xi^2\right) \tag{29}$$

Case 3. When ω_1 is real and ω_2 is purely imaginary, then one can set $\omega_1 = \pm\beta_n/h_1$ and $\omega_2 = \pm i\gamma_n/h_2$ where again, γ_n and β_n are real and positive numbers. In this case equation 27 becomes:

$$A\gamma_n \tan\gamma_n - \beta_n \tanh\beta_n = 0 \tag{30}$$

where

$$\beta_n = \sqrt{h_1^2\left[\xi^2 - \frac{\alpha_2}{\alpha_1}\left(\frac{\gamma^2{}_n}{h_2^2} + \xi^2\right)\right]}$$

Equation 30 usually has a limited number of zeros.

Case 4. When ω_1 is purely imaginary and ω_2 is real, then let $\omega_1 = \pm i\beta_n/h_1$ and $\omega_2 = \pm\gamma_n/h_2$, where, γ_n and β_n are both real and positive. Here equation 27 may be written

$$A\gamma_n \tanh\gamma_n - \beta_n \tan\beta_n = 0 \tag{31}$$

where

$$\beta_n = \sqrt{h_1^2 \left[\frac{\alpha_2}{\alpha_1} \left(\xi^2 - \frac{\gamma_n^2}{h_2^2} \right) - \xi^2 \right]}$$

Equation 31 also has a limited number of zeros.

Depending on the parameters of the problem, zeros of either one or two of the last three cases described above should be considered. Once the zeros are found, corresponding terms in the summation in equation 26 can easily be calculated. In equation 26 the term corresponding to p = 0 is

$$f(\xi) \equiv \frac{(p - 0)P(0)}{g(0)} \tag{32}$$

$$= \frac{\cosh[\xi(z - h_1)]}{\xi^2} \; \frac{K_2\sinh(\xi h_2)\cosh[\xi(h_1 - \ell)] + K_1\cosh(\xi h_2)\sinh[\xi(h_1 - \ell)]}{K_2\sinh(\xi h_2)\cosh(\xi h_1) + K_1\cosh(\xi h_2)\sinh(\xi h_1)}$$

and the term corresponding to $p = -\xi^2\alpha_1$ is

$$\left. \frac{(p + \xi^2\alpha_1)P(p)}{g(p)} \, e^{-\xi^2\alpha_1 t} \right|_{p = -\xi^2\alpha_1} = -\frac{1}{\xi^2} \, e^{-\xi^2\alpha_1 t} \tag{33}$$

Therefore, equation (26) may be written as:

$$G(t) = f(\xi) - \frac{1}{\xi^2} e^{-\xi^2\alpha_1 t} + \sum_{n=1}^{\infty} \frac{P(p_n)}{g'(p_n)} e^{p_n t}$$

where p_n are now only roots of equation 27.

Noting that

$$L^{-1} \left\{ \frac{1}{p\omega_1^2} \right\} = \frac{1}{\xi^2} \left(1 - e^{-\alpha_1 \xi^2 t} \right) \tag{35}$$

the inverse Laplace transform of equation 22 may now be written

$$s_1(t) = L^{-1}\left\{\bar{s}_1(p)\right\}$$

$$= \frac{Q}{2\pi\ell K_1} \int_0^{\infty} J_0(\xi r)\xi \left(\frac{1}{\xi^2} - f(\xi) - \sum_{n=1}^{\infty} \frac{P(p_n)}{g'(p_n)} \; e^{p_n t} \right) d\xi \qquad (36)$$

Introducing the following dimensionless parameters: $S_D = 4\pi K_1 h_1 s/Q$, $t_D = \alpha_1 t/r^2$, $r_D = r/h_1$, $z_D = z/h_1$, $\ell_D = \ell/h_1$, $H = h_2/h_1$, $D = \alpha_2/\alpha_1$, $A = K_2h_1/K_1h_2$, and $x = \xi h_1$, equation 36 becomes

$$s_{D_1} = \frac{2}{\ell_D} \int_0^{\infty} xJ_0(xr_D) \left[\frac{1}{x^2} - f_1(x) + \sum_{n=1}^{\infty} \frac{B_1'}{A'} \exp\left\{ - \frac{\gamma_n^2}{H^2} + x^2 \quad Dr_D^2 t_D \right\} \right] dx \qquad (37)$$

where

$$f_1(x) = \frac{\cosh[x(1 - z_D)]}{x^2} \; \frac{AH\tanh(Hx)\cosh[x(1 - \ell_D)] + \sinh[x(1 - \ell_D)]}{AH\tanh(Hx)\cosh(x) + \sinh(x)} \qquad (38)$$

and the expressions for A' and B_1' depend on the nature of ω_1 and ω_2. If both ω_1 and ω_2 are imaginary, then

$$A' = \left(\frac{1}{2}\right)\left(\frac{\lambda_n^2}{H^2} + x^2\right)\left[\beta_n\left\{D - H^2\left(\frac{\beta_n}{\gamma_n}\right)^2\right\}\cos\gamma_n\sin\beta_n + \beta_n^2\,(AH^2 + D)\cos\gamma_n\cos\beta_n \right.$$

$$\left. - \left(H^2\frac{\beta_n}{\gamma_n} + AD\frac{\gamma_n}{\beta_n}\right)\beta_n^2\,\sin\gamma_n\sin\beta_n \right] \qquad (39)$$

$$B_1' = \cos\left[\beta_n(z_D - 1)\right]\left\{A\gamma_n\sin\gamma_n\cos\left[\beta_n(1 - \ell_D)\right] + \beta_n\cos\gamma_n\sin\left[\beta_n(1 - \ell_D)\right]\right\} \qquad (40)$$

If either ω_1 or ω_2 becomes real, then β_n or γ_n in equations 39 and 40 should be replaced by $(i\beta_n)$ or $(i\gamma_n)$, respectively.

One can find the inversion of $\bar{s}_2$ and $\bar{s}_3$ in a similar manner. In dimensionless forms, the solutions become:

$$s_{D_2} = \frac{2}{\ell_D} \int_0^\infty xJ_0(xr_D)\left[f_2(x) + \sum_{n=1}^{\infty} \frac{B'_2}{A'} \exp\left\{-\left(\frac{\gamma_n^2}{H^2} + x^2\right)Dr_D^2 t_D\right\}\right]dx \quad (41)$$

$$s_{D_3} = \frac{2}{\ell_D} \int_0^\infty xJ_0(xr_D)\left[f_3(x) + \sum_{n=1}^{\infty} \frac{B'_3}{A'} \exp\left\{-\left(\frac{\gamma_n^2}{H^2} + x^2\right)Dr_D^2 t_D\right\}\right]dx \quad (42)$$

where,

$$f_2(x) = \frac{1}{x^2} \; \frac{\cosh[x(z_D + H)]\sinh(x\ell_D)}{AH \sinh(xH)\cosh(x) + \cosh(xH)\sinh(x)} \quad (43)$$

$$f_3(x) = \frac{\sinh(x\ell_D)}{x^2} \; \frac{AH \sinh(xH)\sinh(xz_D) + \cosh(xH)\cosh(xz_D)}{AH \sinh(xH)\cosh(x) + \cosh(xH)\sinh(x)} \quad (44)$$

and when ω_1 and ω_2 are both imaginary,

$$B'_2 = -\beta_n \sin(\beta_n \ell_D)\cos\left[\gamma_n\left(1 + \frac{z_D}{H}\right)\right] \quad (45)$$

$$B'_3 = \sin(\beta_n \ell_D)\left\{A\gamma_n \sin\gamma_n \sin(\beta_n z_D) - \beta_n \cos\gamma_n \cos(\beta_n z_D)\right\} \quad (46)$$

Here, too, if either ω_1 or ω_2 becomes real, then β_n or γ_n in equations 45 and 46 should be replaced by $(i\beta_n)$ or $(i\gamma_n)$, respectively.

Solution for single layer case

The solution for a single layer aquifer with a partially penetrating well can be obtained from the two layer solution by letting the permeability of the lower layer vanish. Letting $K_2 = 0$, equation 22 becomes

$$\bar{S}_1 = \frac{Q}{2\pi \ell K_1} \int_0^\infty \left\{ \frac{1}{P\omega_1^2} - \frac{\cosh[\omega_1(z - h_1)]}{p\omega_1^2} \frac{\sinh[\omega_1(h_1 - \ell)]}{\sinh(\omega_1 h_1)} \right\} \xi J_0(\xi r) d\xi \tag{47}$$

Using the table of the Laplace transformation, one can find

$$L^{-1} \left\{ \frac{\sinh[\omega_1(z - h_1)]}{\omega_1} \frac{\sinh[\omega_1(h_1 - \ell)]}{\sinh(\omega_1 h_1)} \right\} =$$

$$\frac{2\alpha_1}{h_1} \sum_{n=1}^{\infty} \exp\left[-\alpha_1 t \left(\frac{n^2\pi^2}{h_1^2} + \xi^2\right)\right] \sin \frac{n\pi(z - h_1)}{h_1} \sin \frac{n\pi\ell}{h_1} \tag{48}$$

Integrating equation 48 with respect to z gives

$$L^{-1} \left\{ \frac{\cosh[\omega_1(z - h_1)]}{\omega_1^2} \frac{\sinh[\omega_1(h_1 - \ell)]}{\sinh(\omega_1 h_1)} \right\} =$$

$$- \frac{2\alpha_1}{\pi} \sum_{n=1}^{\infty} \exp\left[-\alpha_1 t \left(\frac{n^2\pi^2}{h_1^2} + \xi^2\right)\right] \cdot \frac{1}{n} \sin \frac{n\pi\ell}{h_1} \cos \frac{n\pi(z - h_1)}{h_1} \tag{49}$$

Also,

$$L^{-1} \left\{ \frac{\cosh[\omega_1(z - h_1)]}{p\omega_1^2} \frac{\sinh[\omega_1(h_1 - \ell)]}{\sinh(\omega_1 h_1)} \right\} =$$

$$- \frac{2\alpha_1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi\ell}{h_1} \cos \frac{n\pi(z - h_1)}{h_1} \int_0^t \exp\left[-\alpha_1 \tau \left(\frac{n^2\pi^2}{h_1^2} + \xi^2\right)\right] d\tau \tag{50}$$

By introducing the following change of variables:

$$y = \frac{r^2}{4\alpha_1 \tau} \qquad \text{and} \qquad u = \frac{r^2}{4\alpha_1 t}$$

and finally, the Laplace inversion of equation 47 may be written

$$s_1 = \frac{Q}{4\pi K_1 h_1} \left[\int_u^\infty \frac{e^{-y}}{y}\, dy + \frac{2h_1}{\pi \ell} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi\ell}{h_1} \cos \frac{n\pi(z - h_1)}{h_1} \int_u^\infty \exp\left\{ -y - \frac{(rn\pi)^2}{4yh_1^2} \right\} \frac{dy}{y} \right] \quad (51)$$

Equation 51 is exactly the same as that given by Hantush (1957), and we have been able to check the validity of the new solution for one special case.

Solution for small values of time

To find a solution for the early stages of pumping one has to look for sufficiently large values of p corresponding to small values of t. Let us consider the second part of the integrand in equation 22. Rearranging this term one gets

$$\frac{\cosh[\omega_1(z - h_1)]}{p\omega_1^2} \, \frac{K_2\omega_2 \sinh(\omega_2 h_2)\cosh[\omega_1(h_1 - \ell)] + K_1\omega_1\cosh(\omega_2 h_2)\sinh[\omega_1(h_1 - \ell)]}{K_2\omega_2\sinh(\omega_2 h_2)\cosh(\omega_1 h_1) + K_1\omega_1\cosh(\omega_2 h_2)\sinh(\omega_1 h_1)}$$

$$= \frac{\cosh[\omega_1(z - h_1)]}{p\omega_1^2} \cdot \frac{\sinh[\omega_1(h_1 - \ell)]}{\sinh(\omega_1 h_1)} \cdot \frac{K_2\omega_2\tanh(\omega_2 h_2)\coth[\omega_1(h_1 - \ell)] + K_1\omega_1}{K_2\omega_2\tanh(\omega_2 h_2)\coth(\omega_1 h_1) + K_1\omega_1} \quad (52)$$

Noting that both tanh(x) and coth(x) are almost equal to unity for all values of x greater than 3, the right hand side of equation (52) may be simplified to

$$\frac{\cosh[\omega_1(z - h_1)]}{p\omega_1^2} \; \frac{\sinh[\omega_1(h_1 - \ell)]}{\sinh(\omega_1 h_1)}$$

provided that $\omega_2 h_2 \geqslant 10$ and $\omega_1(h_1 - \ell) \geqslant 10$. As a result, under this condition, equation 22 may be written as

$$\bar{s}_1 = \frac{Q}{2\pi\ell K_1} \int_0^\infty \left\{ \frac{1}{p\omega_1^2} - \frac{\cosh[\omega_1(z - h_1)]}{p\omega_1^2} \; \frac{\sinh[\omega_1(h_1 - \ell)]}{\sinh(\omega_1 h_1)} \right\} \xi J_0(\xi r) \, d\xi$$

which is the same as equation 47 which leads to the solution for the single layer partial penetration problem.

The above conditions may be expressed in terms of dimensionless time. Recalling the definition of ω_2, we can write

$$h_2^2 \left(\frac{p}{\alpha_2} + \xi^2 \right) \geqslant h_2^2 \; \frac{p}{\alpha_2} \geqslant 10 \quad \text{or} \quad \frac{h_2^2}{t\alpha_2} \geqslant 10$$

In terms of dimensionless parameters, this becomes:

$$t_D \leqslant \frac{H^2}{10Dr_D^2} \tag{53}$$

Similarly, the corresponding condition for $\omega_1(h_1 - \ell) \geqslant 10$ leads to:

$$t_D \leqslant \frac{(1 - \ell_D)^2}{10r_D^2} \tag{54}$$

Condition (53) or (54), whichever is smaller, gives an approximate value of t_D. At earlier times, the aquifer behaves as if the lower layer were absent.

Solution for large values of time

To obtain a solution for large values of time, we shall examine the case when p is small. One may note that at large values of time, provided $r_D > 1$, only small values of ξ make a major contribution. Since $\sinh x \approx x$ and $\cosh x \approx 1$ when $x < 0.01$, for sufficiently large values of time and $r > h_1$, equation 22 may be simplified to:

$$\bar{s}_1 = \frac{Q}{2\pi \ell K_1} \int_0^\infty \left\{ \frac{1}{p\omega_1^2} - \frac{1}{p\omega_1^2} \cdot \frac{h_2K_2\omega_2^2 + K_1\omega_1^2(h_1 - \ell)}{h_2K_2\omega_2^2 + K_1\omega_1^2 h_1} \right\} J_0(\xi r)\xi \, d\xi \tag{55}$$

After simplification equation 55 becomes

$$\bar{s}_1 = \frac{Q}{2\pi K_1 h_1} \int_0^\infty \frac{1}{p} \frac{J_0(\xi r)\xi \, d\xi}{\left(\frac{h_2K_2}{h_1K_1}\right)\left(\frac{p}{\alpha_2} + \xi^2\right) + \left(\frac{p}{\alpha_1} + \xi^2\right)} \tag{56}$$

Using the table of Laplace transformations one can easily find

$$s_1 = \frac{Q}{2\pi(K_1h_1 + K_2h_2)} \int_0^\infty J_0(\xi r)\xi \left\{ \frac{1 - \exp\left[-\frac{h_1K_1 + h_2K_2}{S_1 + S_2}\xi^2 t\right]}{\xi^2} \right\} d\xi \tag{57}$$

Equation (57) may be written as (Javandel, 1979),

$$s_1 = \frac{Q}{4\pi(K_1h_1 + K_2h_2)} \int_v^\infty \frac{e^{-y}}{y} \, dy \tag{58}$$

where

$$v = \frac{r^2(S_1 + S_2)}{4t(T_1 + T_2)}$$

In the form of dimensionless parameters we have

$$s_{D_1} = \frac{1}{1 + \frac{T_2}{T_1}} \int_v^\infty \frac{e^{-y}}{y} dy \qquad (59)$$

Since we are dealing with large values of time, equation 59 may be approximated by

$$s_{D_1} \approx \frac{2.3}{1 + (T_2/T_1)} \left(\log t_D + \log \frac{2.25(1 + T_2/T_1)}{1 + S_2/S_1} \right) \qquad (60)$$

This is a very interesting result because it indicates that a plot of dimensionless drawdown, S_{D_1}, versus dimensionless time on semilogarithmic paper will become a straight line when the pumping time becomes sufficiently large. The slope of this line is

$$m = \frac{2.3}{1 + T_2/T_1} \qquad (61)$$

and the value of t_D corresponding to $s_{D_1} = 0$ is

$$t_{D_0} = \frac{1 + S_2/S_1}{2.25(1 + T_2/T_1)} \qquad (62)$$

provided $r_D > 1$. Although equation 62 holds for $r_D > 1$, (61) is true for all values of r_D. Another important result that one be drawn from equation 58 is that, if we introduce a new set of dimensionless definitions for drawdown and time in the following form:

$$\tilde{s}_{D_1} = \frac{4\pi(T_1 + T_2)}{Q} s_1 \qquad (63)$$

and

$$\tilde{t}_D = \frac{t(T_1 + T_2)}{r^2(S_1 + S_2)} \qquad (64)$$

then plots of $\tilde{s}_{D_1}$ versus $\tilde{t}_D$ for two-layer aquifers at large values of time will be parallel to the Theis curve.

WELL OPEN IN THE LOWER LAYER

When the pumping well is open along the length ℓ at the upper part of the lower layer (Fig. 2), then following the same approach as above, one can easily find the solution in the Laplace transform domain of each of the three divisions as given below.

$$\bar{s}_1 = \frac{Q}{2\pi\ell K_2} \int_0^{\infty} \left\{ \frac{\cosh[\omega_1(z - h_1)]}{p\omega_2^2} \cdot \frac{K_2\omega_2 \left\{\sinh\omega_2 h_2 - \sinh[\omega_2(h_2 - \ell)]\right\}}{FF(\omega_1, \omega_2)} \right\} J_0(\xi r)\ \xi\ d\xi \qquad (65)$$

$$\bar{s}_2 = \frac{Q}{2\pi\ell K_2} \int_0^{\infty} \left\{ \frac{1}{p\omega_2^2} - \frac{1}{p\omega_2^2} \cdot \frac{\sinh[\omega_2(h_2-\ell)]\,[K_2\omega_2\cosh\omega_1 h_1\cosh\omega_2 z - K_1\omega_1\sinh\omega_2 z\sinh\omega_1 h_1] + K_1\omega_1\sinh\omega_1 h_1\cosh[\omega_2(z+h_2)]}{FF(\omega_1, \omega_2)} \right\} \cdot J_0(\xi r)\ \xi\ d\xi \qquad (66)$$

$$\bar{s}_3 = \frac{Q}{2\pi\ell K_2} \int_0^{\infty} \left\{ \frac{\cosh[\omega_2(z + h_2)]}{p\omega_2^2} \cdot \frac{-K_1\omega_1\sinh\omega_1 h_1 + K_1\omega_1\sinh\omega_1 h_1\cosh\omega_2\ell + K_2\omega_2\cosh\omega_1 h_1\sinh\omega_2\ell}{FF(\omega_1, \omega_2)} \right\} \cdot J_0(\xi r)\ \xi\ d\xi \qquad (67)$$

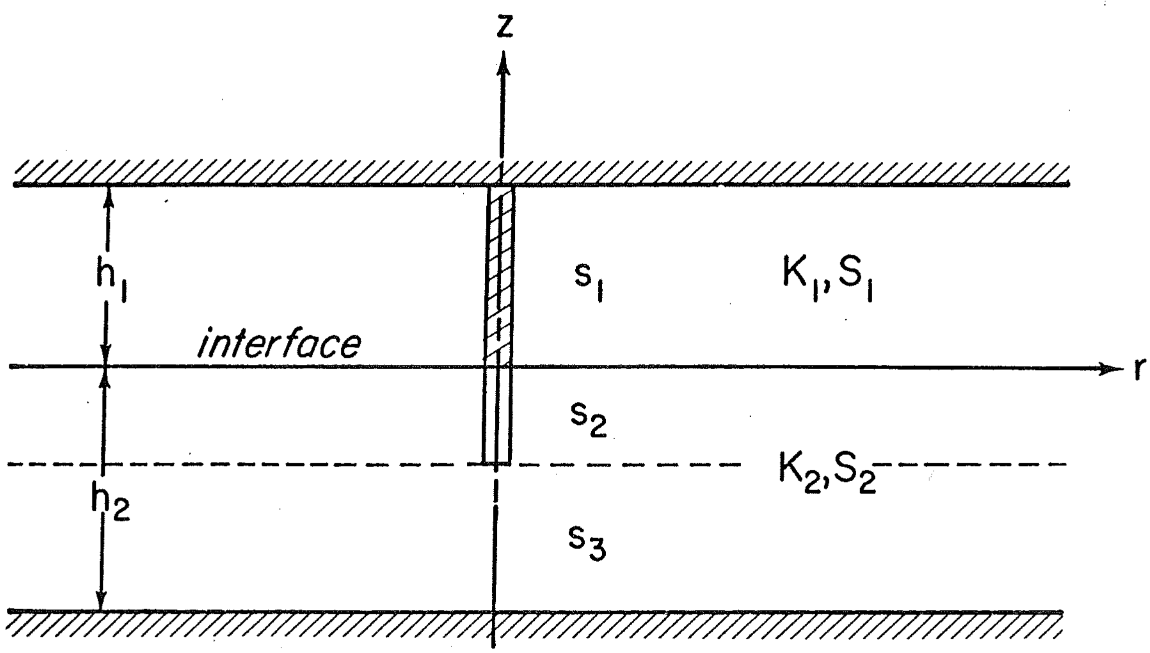

Figure 2. Schematic diagram of a two-layered aquifer with a partially penetrating well open only in the lower layer.

It can be easily verified that equations 65 through 67 satisfy the appropriate boundary conditions of the problem. The Laplace inversion of the above expressions can be readily obtained through the same procedure discussed above. In regard to the inversion of $\bar{s}_1$, for example, note that the nonremovable zeros of the denominator of the integrand are $p = 0$ as well as all the roots of $FF(\omega_1, \omega_2) = 0$, which have already been discussed above. Finally, the solutions for s_1, s_2, and s_3 in the nondimensional form are

$$s_{D_1} = \frac{2}{\hat{\ell}_D} \int_0^\infty \hat{x} J_0(\hat{x}\hat{r}_D) \left\{ R_1(\hat{x}) + \sum_{n=1}^{\infty} \frac{q_1}{\delta} \exp\left[-(\gamma_n^2 + \hat{x}^2)\hat{t}_D \hat{r}_D^2\right] \right\} d\hat{x} \qquad (68)$$

$$s_{D_2} = \frac{2}{\hat{\ell}_D} \int_0^\infty \hat{x} J_0(\hat{x}\hat{r}_D) \left\{ \frac{1}{\hat{x}^2} - R_2(\hat{x}) + \sum_{n=1}^{\infty} \frac{q_2}{\delta} \exp\left[-(\gamma_n^2 + \hat{x}^2)\hat{t}_D \hat{r}_D^2 \right] \right\} d\hat{x} \quad (69)$$

$$s_{D_3} = \frac{2}{\hat{\ell}_D} \int_0^\infty \hat{x} J_0(\hat{x}\hat{r}_D) \left\{ R_3(\hat{x}) + \sum_{n=1}^{\infty} \frac{q_3}{\delta} \exp\left[-(\gamma_n^2 + \hat{x}^2)\hat{t}_D \hat{r}_D^2 \right] \right\} d\hat{x} \quad (70)$$

where

$$R_1(\hat{x}) = \frac{\cosh[\hat{x}(\hat{z}_D - \hat{H})]}{\hat{x}^2} \cdot \frac{\sinh(\hat{x}) - \sinh[\hat{x}(1 - \hat{\ell}_D)]}{\cosh(\hat{x}\hat{H})\sinh(\hat{x}) + (\hat{H}/A)\sinh(\hat{x}\hat{H})\cosh(\hat{x})} \quad (71)$$

$$R_2(\hat{x}) = \frac{1}{\hat{x}^2} \cdot \quad (72)$$

$$\frac{\sinh[\hat{x}(1 - \hat{\ell}_D)][\cosh(\hat{x}\hat{H})\cosh(\hat{x}\hat{z}_D) - (\hat{H}/A)\sinh(\hat{x}\hat{z}_D)\sinh(\hat{x}\hat{H})] + (\hat{H}/A)\sinh(\hat{x}\hat{H})\cosh[\hat{x}(\hat{z}_D + 1)]}{\cosh(\hat{x}\hat{H})\sinh(\hat{x}) + (\hat{H}/A)\sinh(\hat{X}\hat{H})\cosh(\hat{x})}$$

$$R_3(\hat{x}) = \frac{\cosh[\hat{x}(\hat{z}_D + 1)]}{\hat{x}^2} \cdot$$

$$\cdot \frac{-(\hat{H}/A)\sinh(\hat{x}\hat{H}) + (\hat{H}/A)\sinh(\hat{x}\hat{H})\cosh(\hat{x}\hat{\ell}_D) + \cosh(\hat{x}\hat{H})\sinh(\hat{x}\hat{\ell}_D)}{\cosh(\hat{x}\hat{H})\sinh(x) + (\hat{H}/A)\sinh(\hat{x}\hat{H})\cosh(\hat{x})} \quad (73)$$

and when ω_1 and ω_2 are both imaginary,

$$q_1 = -\gamma_n \cos\left[\beta_n\left(\frac{\hat{z}_D}{\hat{H}} - 1\right)\right] \left\{ \sin\gamma_n - \sin\left[\gamma_n(1 - \hat{\ell}_D)\right] \right\} \quad (74)$$

$$q_2 = \sin[\gamma_n(1 - \hat{\ell}_D)] \left\{ \gamma_n \cos\beta_n \cos(\gamma_n \hat{z}_D) + (\beta_n/A)\sin(\gamma_n \hat{z}_D)\sin\beta_n \right\}$$

$$+ \quad (\beta_n/A)\sin\beta_n \cos[\gamma_n(\hat{z}_D + 1)] \tag{75}$$

$$q_3 = \cos[\gamma_n(\hat{z}_D + 1)] \left\{ (\beta_n/A)\sin\beta_n - (\beta_n/A)\sin\beta_n \cos(\gamma_n \hat{\ell}_D) - \gamma_n \cos\beta_n \sin(\gamma_n \hat{\ell}_D) \right\} \tag{76}$$

$$o = \frac{1}{2}(\gamma_n^2 + \hat{x}^2) \left\{ \beta_n \sin\beta_n \cos\gamma_n \left(\frac{D\hat{H}^2 \gamma_n^2}{A\beta_n^2} - \frac{1}{A} \right) - \gamma_n^2 \sin\beta_n \sin\gamma_n \left(\frac{D\hat{H}^2 \gamma_n}{\beta_n} + \frac{\beta_n}{A\gamma_n} \right) \right.$$

$$\left. + \quad \gamma_n^2 \cos\beta_n \cos\gamma_n \left(1 + \frac{D\hat{H}^2}{A} \right) \right\} \tag{77}$$

If either ω_1 or ω_2 becomes real then β_n or γ_n in equations 74 through 77 should be replaced by $(i\beta_n)$ or $(i\gamma_n)$, respectively.

Examination of equation 66 reveals that if we let the permeability of the top layer vanish, the solution for s_2 converges to the one for single-layer partial penetration. However, in this case, due to the direct contact with the top layer, the solution at small values of time cannot be closely approximated with single layer solutions unless the ratio of K_1/K_2 is very small.

CONCLUSIONS

Analytic solution to the problem of transient flow toward a partially penetrating well in a two-layered aquifer has been presented. A solution has been given for both cases: when the well is open in the upper layer as well as the case when the well is open in the lower one. These solutions easily lend themselves to numerical evaluation. It has been shown that the solutions would reduce to the case of single layer partial penetration once we allow the permeability of the nonperforated layer to vanish. Asymptotic solutions for small and large values of time have been deduced from the transformed form of solution. Furthermore, it was shown that:

(1) the behavior of the pumped layer at early times is exactly similar to the behavior of a single-layer aquifer;

(2) for larger values of time the plot of dimensionless drawdown s_{D_1} versus dimensionless time t_D on a semilogarithmic paper becomes a straight line whose slope is only a function of the ratio of transmissibility of the two layers.

NOTATION

Symbol	Definition	Dimensions
A	K_2h_1/K_1h_2	--
D	α_2/α_1	--
h_1	thickness of the top layer	L
h_2	thickness of the lower layer	L
H	h_2/h_1	--
$\hat{H}$	h_1/h_2	--
$J_0(x)$	Bessel's function of the first kind and zero order	--
K_1, K_2	permeability of upper and lower layers, respectively	L/T
ℓ	depth of penetration	L
ℓ_D	ℓ/h_1	--
$\hat{\ell}_D$	ℓ/h_2	--
L^{-1}	Laplace transform inversion operator	--
p	Laplace transform parameter	T^{-1}
Q	rate of discharge	L^3/T
r	radial distance	L
r_D	r/h_1	--
$\hat{r}_D$	r/h_2	--
s_i	drawdown of different layers	L
s_{D_i}	$4\pi K_1h_1s_i/Q$, dimensionless drawdown	--
$\hat{s}_{D_i}$	$4\pi K_2h_2s_i/Q$, dimensionless drawdown	--

$\tilde{s}_{D_1}$	$4\pi(T_1 + T_2)s_1/Q$	--
S_1, S_2	storage coefficient of the upper and lower layer, respectively	
$\bar{s}_i$	Laplace transform of s_i	L
$\bar{\bar{s}}_i$	Hankel transform of $\bar{s}_i$	
t	time	T
t_D	$\alpha_1 t/r^2$, dimensionless time	--
$\hat{t}_D$	$\alpha_2 t/r^2$, dimensionless time	--
$\tilde{t}_D$	$t(T_1 + T_2)/r^2(S_1 + S_2)$, dimensionless time	--
T_1, T_2	transmissibility of the upper and lower layer, respectively	L^2/T
x	ξh_1, dummy variable	--
$\hat{x}$	ξh_2, dummy variable	--
z	vertical coordinate	L
z_D	z/h_1	--
$\hat{z}$	z/h_2	--
α_1, α_2	diffusivity of layer 1 and 2, respectively	L^2/T
b_n	$h_1^2[a_2/a_1(q_n^2/h_2^2 + u^2) - u^2]^{1/2}$	--
γ_n	roots of characteristic equation 28	--
ξ	Hankel transform parameter	L^{-1}
ω_1	$i\beta/h_1$	--
ω_2	$i\gamma/h_2$	--

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