GL04154

GEOPHYSICS, VOL. 48, NO. 9 (SEPTEMBER 1983); P. 1269–1273, 4 FIGS.

Short Note

Applications of bidimensional spline functions to geophysics

José Oscar Campos Enriquez*, Jean Thomann‡, and Michel Goupillot*

INTRODUCTION

Two bidimensional spline functions are applied to the interpolation and automatic contouring of data from a ground magnetic survey of the Rhinegraben, between Karlsruhe and Strasbourg. These bidimensional spline functions can be applied in the general case, i.e., where a set of nonequispaced data is given. These splines define a surface passing through the original points. The most promising spline functions are the thin plate ("plaque mince") and the pseudocubic spline, studied by Atteia (1966), Thomann (1970), and Duchon (1975, 1976). A contouring method based on them has been implemented.

A brief description of these bidimensional spline functions is given below, together with some properties and practical examples of their application in automatic contouring of irregularly distributed data.

Finally, their advantage over bicubic spline functions, in interpolating random distributed data, is also discussed.

THIN PLATE AND PSEUDOCUBIC SPLINE FUNCTIONS

The current cubic spline functions, with knots at the points $a_1 < \cdots < a_n$, can be defined in the following forms.

(1) Cubic polynomials in each of the intervals (a_i, a_{i+1}) and affine functions in $(-\infty, a_1)$ and $(a_n, +\infty)$, with their two first derivatives continuous everywhere, i.e., belonging to C^2 (Greville, 1968; Utreras, 1978).

(2) Functions of the form

$$\sum \lambda_i |t - a_i|^3 + \alpha t + \beta$$

with

$$\sum \lambda_i = 0 \quad \text{and} \quad \sum \lambda_i a_i = 0$$

(Greville, 1968).

(3) As a solution to the interpolation problem of minimum energy:

$$\text{Minimize} \quad \int |V''|^2$$

under the conditions $v(a_i) = Z_i$ (Utreras, 1979).

Each definition can be generalized to a bidimensional space. The first definition leads to the well-known bicubic splines (De Boor, 1962). The second definition gives rise to the pseudocubic spline function (Atteia, 1966, 1970). The last one results in the thin plate spline function studied by Atteia (1966), Thomann (1970), and Duchon (1975, 1976).

They are particular cases of the spline functions of order (m, s) which minimize a functional invariant by rotation, that is expressed with the norm of derivatives of order m in a Hilbert space H^s (Duchon, 1976).

For the thin plate spline, $m = 2$ and $s = 0$, and the functional is of the form

$$e(V) = \int_\Omega |\nabla^2 V| = \int_\Omega \left\{ \left| \frac{\partial^2 V}{\partial x^2} \right|^2 + 2 \left| \frac{\partial^2 V}{\partial x \, \partial y} \right|^2 + \left| \frac{\partial^2 V}{\partial y^2} \right|^2 \right\}. \tag{1}$$

This functional represents, in a first approximation, the flection energy of a thin plate occupying a position defined by the function V. The surface obtained employing this spline is similar to the one that would be obtained by deforming a thin elastic plate with pressure applied at each of the points where the function is known. These pressures are proportional to the values of the function.

For the pseudocubic spline function, $m = 2$ and $s = 1/2$. These functions have a probabilistic interpretation: they represent the best estimation of random functions, where the gradient is a "kind of Brownian movement" (Duchon, 1975).

ANALYTICAL EXPRESSION AND NUMERICAL CALCULATION

The analytical expression for these spline functions is

$$\sigma(t) = \sum_{i=1}^{n} \lambda_i K(t^i, t) + \alpha_1 X + \alpha_2 Y + \beta, \tag{2}$$

with

$$\sum_{i=1}^{n} \lambda_i = 0, \tag{3}$$

$$\sum_{i=1}^{n} \lambda_i X_i = 0, \tag{4a}$$

Manuscript received by the Editor December 30, 1980; revised manuscript received December 29, 1982.
*Laboratoire de Paléomagnétisme, Institut de Physique du Globe, Strasbourg, France.
‡Centre de Calcul, C.N.R.S., Strasbourg, France.

and

$$\sum_{i=1}^{n} \lambda_i Y_i = 0, \tag{4b}$$

$$\sigma(t^i) = Z^i. \tag{5}$$

$t^1, \ldots, t^n$ are the points where the values $Z^1, \ldots, Z^n$ were observed, t is a general point of the plane R^2. $K(t^i, t)$ is a reproducing kernel of a Hilbert space (Aronszajn, 1950). For the thin plate spline,

$$K(t^i, t) = |t - t^i|^2 \log |t - t^i|^2 \tag{6}$$

and for the pseudocubic spline

$$K(t^i, t) = |t - t^i|^{3/2}, \tag{7}$$

where $|\cdot|$ is the Euclidean norm.

For a given set of values $Z^1, \ldots, Z^n$ at the points $t^1, \ldots, t^n$, these functions are defined by the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n, \alpha_1, \alpha_2, \beta$. They are obtained by solving the system of equations (3), (4a), (4b), and (5), that is

$$\begin{bmatrix} K & E \\ E^t & 0 \end{bmatrix} \begin{bmatrix} \Lambda \\ \alpha \end{bmatrix} = \begin{bmatrix} Z \\ 0 \end{bmatrix}, \tag{8}$$

where

$$K = [k_{ij}]_{n \times n}, \quad k_{ij} = K(t^i, t^j), \quad k_{ii} = 0$$

$$Z^t = [Z^1, \ldots, Z^n], \quad \alpha^t = [\beta, \alpha_1, \alpha_2], \quad \Lambda^t = [\lambda_1, \lambda_2, \ldots, \lambda_n],$$

and

$$E^t = \begin{bmatrix} 1, & \ldots, & 1 \\ X_1, & \ldots, & X_n \\ Y_1, & \ldots, & Y_n \end{bmatrix}.$$

This system is ill-conditioned and its solution is not straightforward. For the case where Ω is bounded by a circle or a rectangle, Thomann (1970) developed an algorithm to calculate the thin plate spline. Paihua (1978) proposed some algorithms to solve system (8) for the case $\Omega = R^2$.

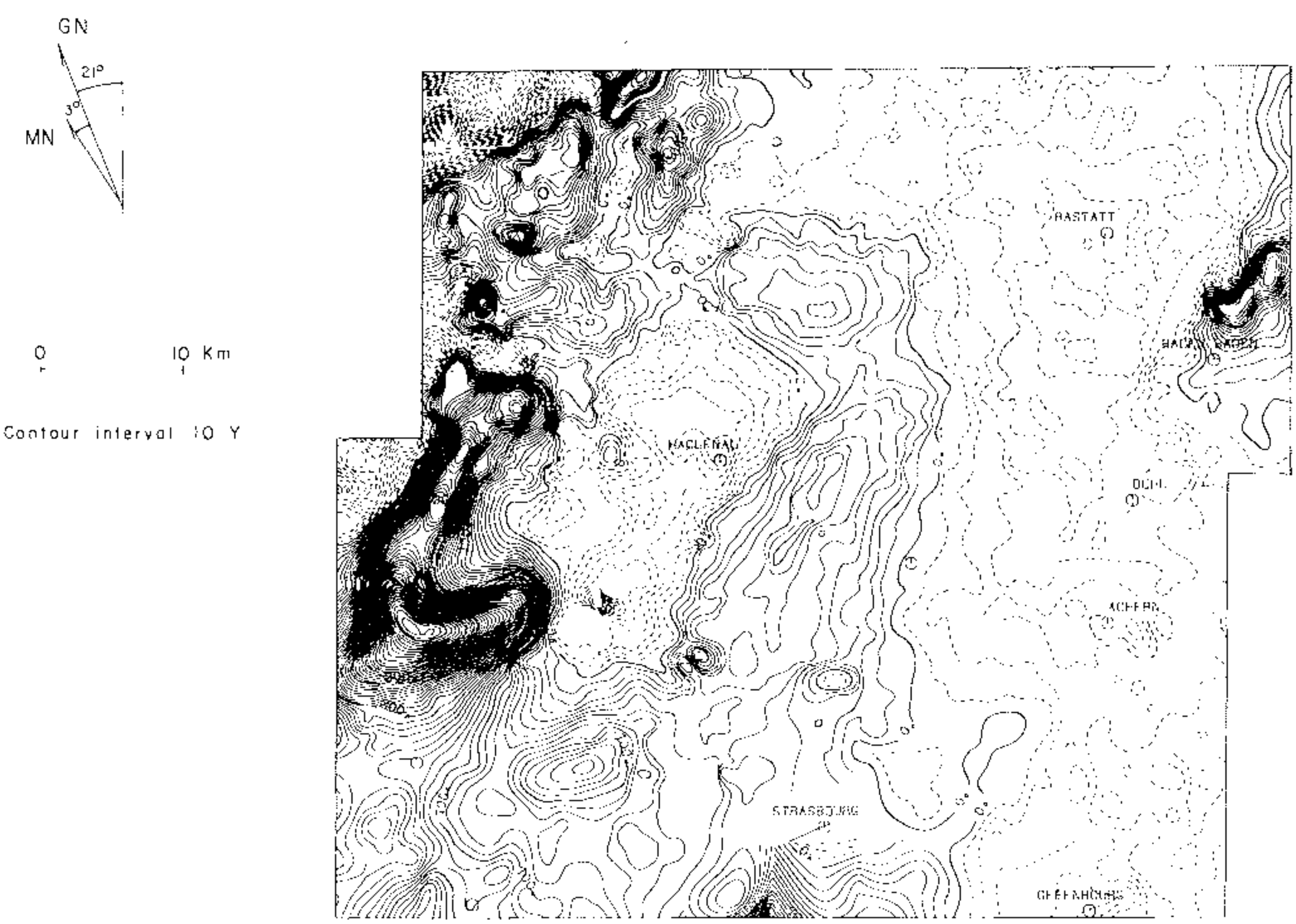

FIG. 1. The map was elaborated using the pseudocubic spline.

SOME PROPERTIES OF THE THIN PLATE SPLINE

We have seen that the solution of the system (8) is conditioned to pass through the original points, i.e., $\sigma(t^i) = Z^i$. We noticed also that this solution directly gives the coefficients that define the spline function. In this solution we only need to use the original data: Z^i at the point t^i, i.e., their application is direct. The thin plate spline belongs to the space of continuous functions on R^2, which first and second derivatives (in the sense of distributions) are in $L^2(R^2)$, i.e., are bounded. The integral (1) represents the sum of squared curvatures, and since expression (1) is a minimum only when V is a thin plate spline, we conclude that the thin plate spline can be viewed as the "smoothest" of all possible interpolating functions.

AUTOMATIC CONTOURING USING BIDIMENSIONAL SPLINES

Thomann (1970) developed the following algorithm.

(1) A partition of the area of interest is done, generally in rectangles containing at most 50 points.

(2) A function for interpolation is then calculated in each cell. This is done by solving system (8). This calculation also considers those points belonging to the surrounding adjacent cells, i.e., in a finite rectangular strip around the cell. This is done to ensure continuity of contours between cells.

(3) To construct the contours, a search is done for the "extremums" in each cell (maximums and minimums). It is done under the hypothesis that outside each cell the interpolation function has very large values, $+\infty$ (very small values, $-\infty$) for the search of minimums (in the search of maximums).

(4) These points are joined by linear segments, from left to right on the same horizontal line, otherwise from bottom to top; this constitutes the exploration path. This is a variation of the method proposed by Holroyd and Bhattacharyya (1970).

(5) We search for the intersections of desired contours with this exploration path.

(6) Each time an intersection is found, we proceed to the

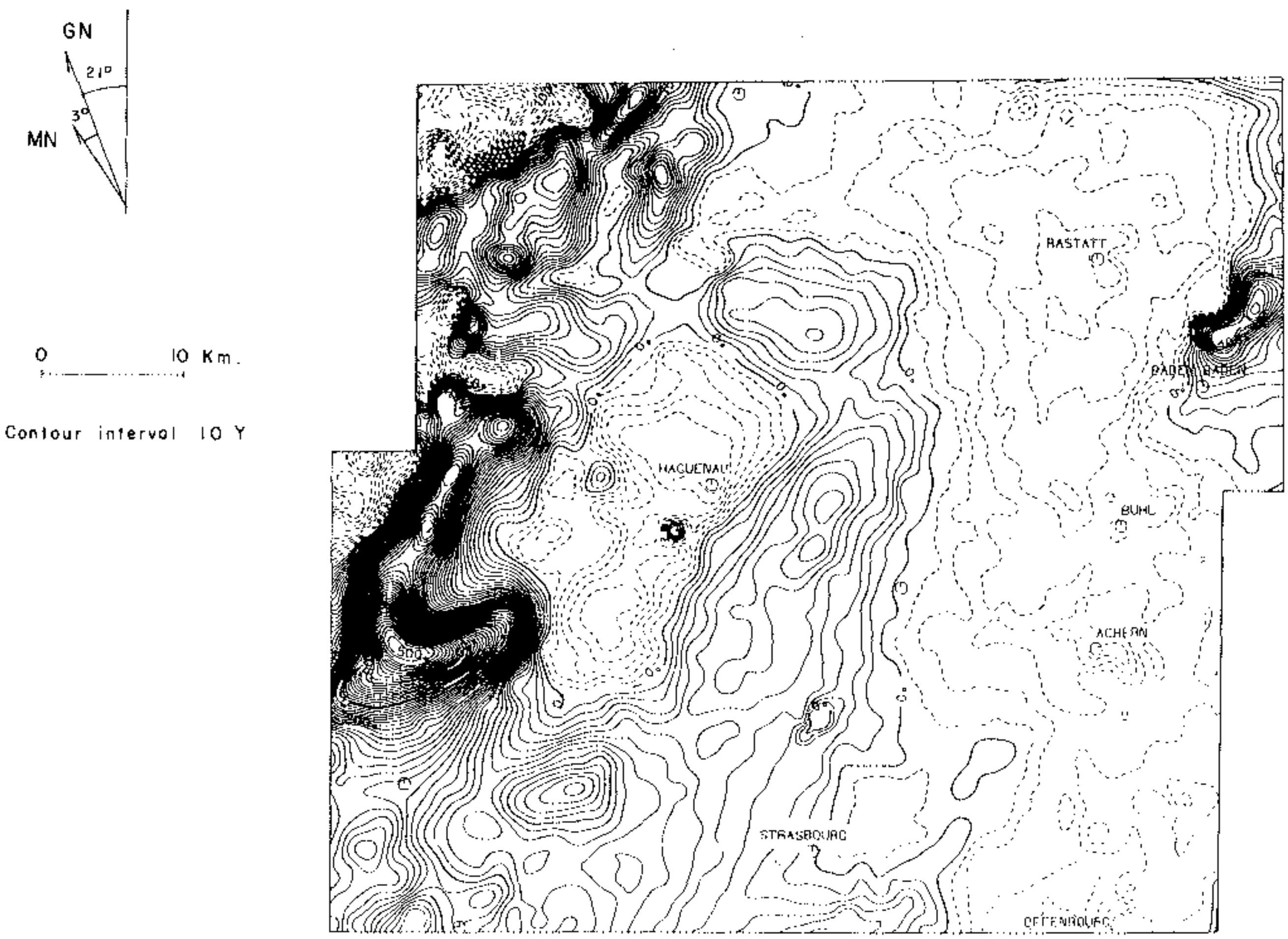

FIG. 2. The contour map of the interpolated data with a 1 km spacing. The spline used was the pseudocubic.

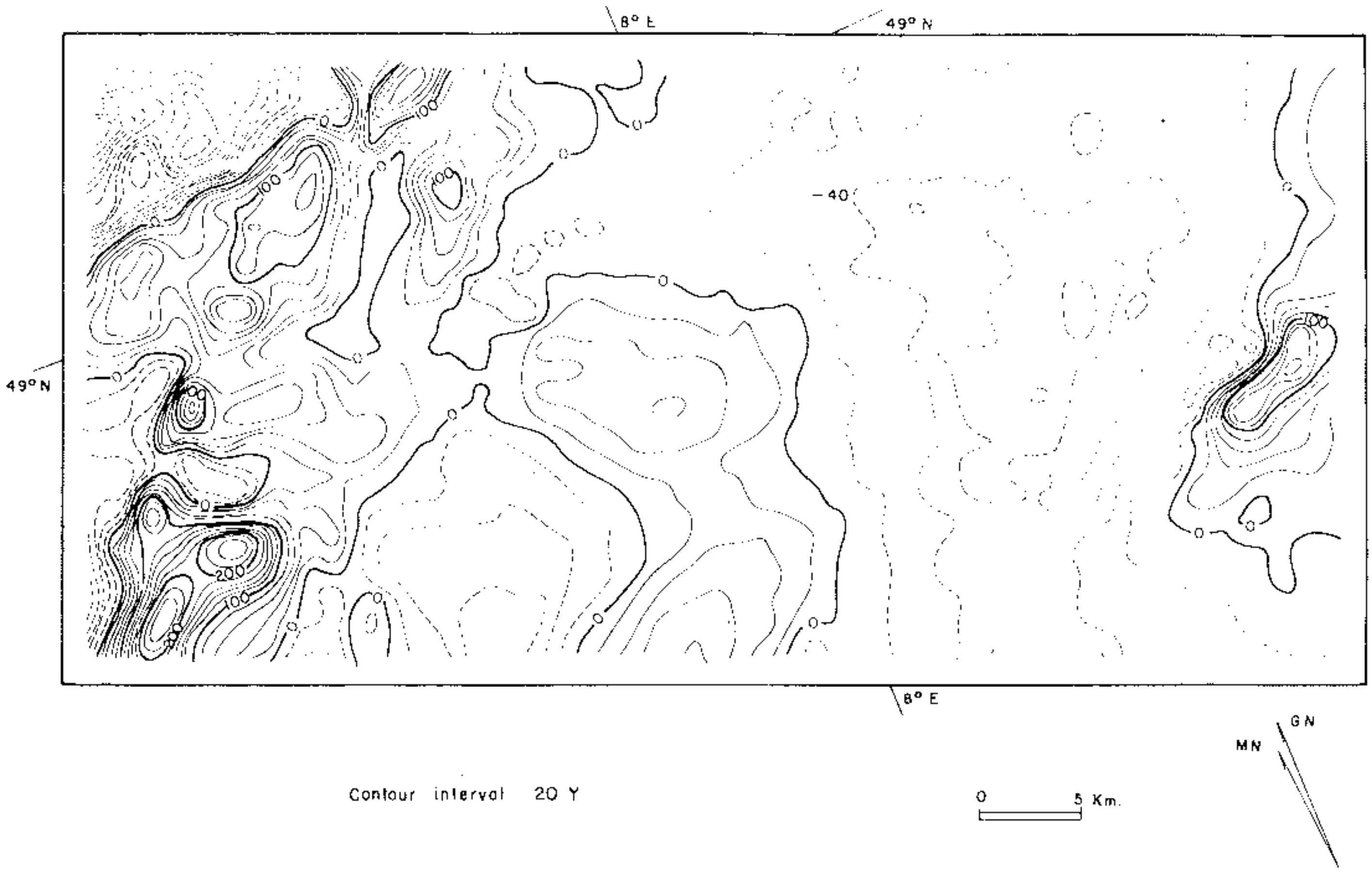

FIG. 3. Contour map of the northern half of the area of this study, after Goupillot (1977). The spline used in its elaboration was the thin plate type.

construction of the contour. Each contour is considered closely. In this way, by using an optimization algorithm, we construct a given contour only once.

This algorithm can be used with either the thin plate spline or the pseudocubic spline.

PRACTICAL EXAMPLES

We have used these splines to construct the contour map of the total magnetic anomaly of the Rhinegraben between Strasbourg and Karlsruhe (Goupillot, 1977; Edel et al, 1980; Campos, 1980).

Figure 1 shows the contour map obtained using the pseudocubic spline. The quality of this map is not as good in those parts where there is a lack of measurements—as for example in the vicinity of Strasbourg as it is elsewhere.

Figure 2 shows the contour map obtained from values interpolated at the intersections of a regular square net, 1 km wide. The interpolation was made using La Porte's method (La Porte, 1962). In this map the anomalies have been somewhat smoothed.

Figure 3 shows the contour map for the northern part of this survey. Goupillot (1977) obtained it using the thin plate spline.

Finally, Figure 4 shows, for the northern part of this zone, the contour map of the upward continuation (height of continuation 600 m above ground level). It was constructed using thin plate splines (Goupillot 1977).

CONCLUSIONS

In brief, thin plate and pseudocubic splines produce maps of high quality: minimum curvature in the case of the thin plate spline; continuity for both spline functions; and for the pseudocubic spline, continuity of its first partial derivatives. These functions are well suited to interpolation of nonequispaced data whatever distribution they have. This is the advantage of these two spline functions over current bicubic splines. Current spline functions give good results only in particular cases, for example with data distributed nearly on straight lines (aeromagnetic data). But even here, because they are not of direct application, i.e., a preliminary interpolation is needed, they can not produce interpolation surfaces passing through the original points.

ACKNOWLEDGMENTS

This work was done in the framework of the contract CEE-CNRS no. 574-78-1 egt. between the Commission of the Euro-

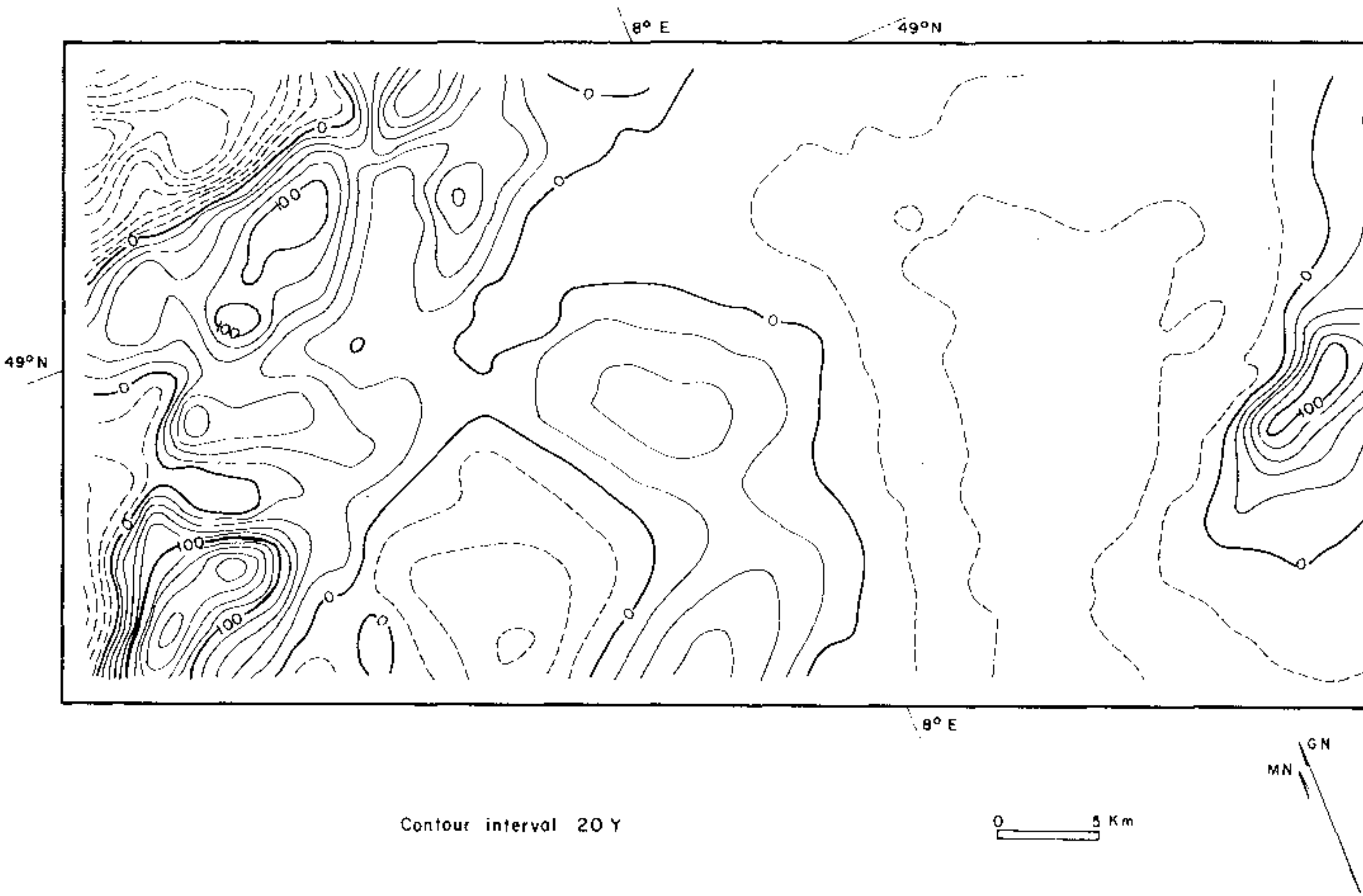

FIG. 4. Contour map of the upward continuation of the data of the northern half area, after Goupillot (1977). The height of continuation is 600 m above ground level. The spline used in its elaboration was the thin plate type.

pean Communities and the Institut de Physique du Globe, Strasbourg, France.

REFERENCES

Atteia, M., 1966, Existence et détermination des fonctions spline à plusieurs variables: C. R. Acad. Sc. Paris, v. 269 série A, p. 575–578.

——— 1970, Fonctions spline et noyaux reproduisants d'Aronszajn-Bergman: Rev. Franc. Inf. Rech. Opér., October-November p. 31-43.

Aronszajn, N. A., 1950, Theory of reproducing Kernels: Trans. Amer. Math. Soc.

Campos Enriquez, J. O., 1980, Elaboration, traitement et interprétation de la carte magnétique du Fossé Rhénan (Wissembourg, Karlsruhe, Offenbourg, Saverne).: Strasbourg, Ins. de Phys. du Globe, Thèse de Docteur-Ingénieur.

De Boor, C., 1962, Bicubic spline interpolation: J. Math. and Phys., v. 41, p. 212–218.

Duchon, J., 1975, Fonctions spline du type plaque mince en dimension 2: Séminaire d'analyse numérique n. 231, U.S.M.G., Grenoble.

——— 1976, Fonctions spline à énergie invariante par rotation: Rapport de recherche n. 27, Math. Appl. U.S.M.G., Grenoble.

Edel, J. B., Roche, A., Campos, J. O., Goupillot, M., Kiro, K. N., Menard, Y., Merheb, F., 1980, Contribution of magnetism and gravimetry to the knowledge of the antepermian basement in the Rhinegraben. Applications to geothermy: Second International seminar on the results of the European Communities Geothermal Energy Research, Strasbourg, March 4, 5, 6.

Goupillot, M., 1977, Prospection magnétique dans le nord du Fossé Rhénan (Haguenau, Wissembourg, Niederbronn, Rastatt), Méthode de traitement de la carte obtenue: Diplôme d'Ingénieur-Géophysicien, Strasbourg, Inst. de Phys. du Globe.

Greville, T. N. E., 1968, Spline functions: in the theory and applications of spline functions, T. N. E. Greville, Ed.: Proceedings of an advanced seminar conducted by the Mathematics Research Center, United States Army, Univ. of Wisconsin, Madison, Oct. 7–9, Academic Press.

Holroyd, M. T., and Bhattacharyya, B. K., 1970, Automatic contouring of geophysics field data using bicubic spline interpolation: Geol. Surv. of Canada, paper 70–75.

La Porte, M., 1962, Elaboration rapide de cartes gravimétriques déduites de l'anomalie de Bouguer à l'aide d'une calculatrice électronique: Geophys. Prosp., v. 10, p. 238–257.

Paihua, L., 1978, Quelques méthodes numériques pour le calcul de fonctions spline a une et plusieurs variables: Thèse de 3 ème cycle, U.S.M.G., Grenoble.

Thomann, J., 1970, Détermination et construction de fonctions spline à deux variables définies sur un domaine rectangulaire ou circulaire: Thèse de 3 ème cycle, Lille.

Utreras Diaz, F., 1978, Utilisation des programmes de calcul du paramètre d'ajustement dans le lissage par fonctions spline: R.R. n. 121, Mai 1978, U.S.M.G., Grenoble.

——— 1979, Utilisation de la méthode de validation croisée pour le lissage par fonctions spline à une ou deux variables: U.S.M.G., Grenoble.